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Uniqueness of stationary equilibria in bargaining one-dimensional policies under (super) majority rules

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ABSTRACT

We consider negotiations selecting one-dimensional policies. Individuals have instantaneous preferences represented by continuous, concave and single-peaked utility functions, and they are impatient. Decisions arise from a bargaining game with random proposers and (super) majority approval, ranging from the simple majority up to unanimity. We provide sufficient conditions that guarantee the existence of a unique stationary subgame perfect equilibrium, and we provide its explicit characterization. The uniqueness of the equilibrium permits an analysis of the set of Pareto optimal voting rules. For symmetric distributions of peaks and uniform recognition probabilities unanimity is the unanimously preferred majority rule.

1. Introduction

Collective decisions in democratic polities are often the result of processes where bargaining and majority voting are combined. This paper examines this type of negotiations under the assumption that policies must be selected from a continuous one-dimensional set, where individuals have instantaneous preferences represented by continuous, concave and single-peaked utility functions. This simple set-up is a classical formulation in the social choice and political economy literature. Examples are the location of a facility, the election of a public official, the choice of tax rates or minimum wages, or the budget allocated to a specific project. In this paper, we assume that individuals are heterogeneous only in the locations of peaks, that decisions must be negotiated over time (individuals are impatient) under a stationary random proposer protocol, and that the approval of a (super) majority of the group is required for an agreement. Our contribution is to provide its explicit characterization and establish the uniqueness of the equilibrium. Hence, for a natural and tractable formulation, we supply the precise prediction for what alternatives prevail in negotiations under each majority rule. This allows tractable comparative statics, which is the tool to address a wide range of applications. The application that we explore in this paper examines the Pareto efficiency of majority rules.

More precisely, we fully describe the unique stationary equilibrium under the standard random proposers protocol: At the beginning of each round, an agent is selected at random to make a proposal which is approved if it obtains the favorable vote of a (super) majority. Upon approval, the selected alternative is implemented and the game ends. If the proposal is not approved, a new round of bargaining begins in the following period. It is assumed that individuals are impatient. This formulation is a special case of the general model of Banks and Duggan (2000). Therefore the existence of a

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stationary subgame perfect equilibrium,[1] and its no-delay property, follows from their results. Our characterization and proof of uniqueness builds on the following argument. Because equilibria involve no delay, every proposal arising in equilibrium must receive the favorable vote of (at least) a majority q. An individual votes for a proposal only if it lies in her acceptable set, i.e. in the subset of alternatives that are better than delaying play for one period. Moreover, a proposal is approved only if it lies in the approval set, the subset of alternatives that are in the acceptable sets of some majority of size at least q. Therefore, the proposer puts forward her most preferred alternative within the approval set. Characterizing an equilibrium is tantamount to providing the necessary and sufficient conditions to determine the approval set. These conditions require a fixed point: From an interval of alternatives that receive approval, we compute the associated expected payoffs, which determine the individual acceptance sets, and in turn induce an interval of approval. In equilibrium, the later approval set must coincide with the former. Our main results are the precise description of this condition and the observation that under natural conditions it delivers one and only one solution.

The unique equilibrium induces a unique distribution of approved alternatives and unique individual (expected) utilities for each majority rule. Hence, we can evaluate the expected utilities of each individual under different majority requirements, and thus examine what (super) majority rules are Pareto optimal. Under a symmetric distribution of peaks and uniform recognition probabilities very strong results apply: weakening the majority requirement spreads the approval set symmetrically around a preserved mean. For strictly concave utilities, this implies that all individuals have a strict preference for unanimity over any other majority rule. The conclusion is that in symmetric environments unanimity is the unique Pareto optimal super-majority.

The remainder of the paper is organized as follows. Section 2 discusses the related literature. The environment and the bargaining game are presented in Section 3. In Section 4, we characterize the stationary subgame perfect equilibrium and we establish uniqueness. Section 5 examines Pareto optimal rules. Section 6 contains final remarks. Proofs omitted from the main text are in Appendix A.

2. Relation to the literature

The literature that addresses multilateral bargaining over policy choices initiated by Baron and Ferejohn (1989) has mostly focused to situations in which a unit of surplus must be distributed, and decisions require a simple majority. Eraslan (2002) extends the analysis to set-ups with heterogeneous discounts and recognition probabilities (or protocols), and considers the full range of super-majority rules. She establishes the uniqueness of stationary subgame perfect equilibrium payoffs under linear utilities, and shows that expected payoffs are egalitarian under a wide range of asymmetric protocols.[2] Our approach complements this literature by examining the opposite polar case: rather than examining the transfers that are necessary for agreement, we examine negotiations where agreements must lie in an interval, so that transfers of resources among the parties are impossible.[3] Banks and Duggan (2000, 2006) examine a general model that covers both approaches as particular cases. They assume that alternatives are selected from arbitrary compact convex subsets of an Euclidean space, and they examine bargaining protocols where the proposer is selected at random and approval is determined by voting rules in a general family. They prove existence of equilibria under very general conditions, and they establish sufficient conditions for core equivalence. For set-ups where alternatives are in an interval, they show that equilibria (in pure strategies) exist; and that for perfectly patient players they are equivalent to core outcomes. Their results, however, do not provide an explicit characterization of equilibria for impatient players, nor a discussion of conditions for uniqueness. The issue of equilibrium uniqueness for one-dimensional problems is addressed in Cho and Duggan (2003) and Cardona and Ponsati (2007). Cho and Duggan (2003) assume quadratic utilities and consider decisions rules in a family that includes the simple majority, but not stronger super-majorities. For these set-ups, they characterize the equilibrium and establish its uniqueness for games with random proposers. Cardona and Ponsati (2007) assume that negotiations follow a deterministic protocol, and establish the uniqueness of the equilibrium for all majority rules. These results apply to a very rich class of populations, since peaks and utilities can vary across individuals. But this generality comes at the expense of tractability. The characterization of equilibrium is rather involved and, unlike in the present paper, the results are not easily ready for applications. They also prove that, as players become arbitrarily patient, the equilibrium converges to a single alternative, which is independent of the protocol. The asymptotic uniqueness of the equilibrium in games with random proposals is established in Predtetchinski (2011). In particular, the unique asymptotic equilibrium is shown to be dependent on the set of players, the utility functions, the recognitions probabilities and the acceptance rule.

We remark that most of the literature concerns situations where an alternative is selected only once and for all, none can be implemented in disagreement, and disagreement is the worst outcome for everyone. Some important collective decisions are of this nature; for example the location of a public facility or the appointment of public officials. Other policies, such

[1] The restriction to stationary strategies is standard in multilateral bargaining games, as these games are known to have very large sets of subgame perfect equilibria.

[2] The importance of stationarity in yielding a unique prediction has been also stressed by Merlo and Wilson (1995). They provide sufficient conditions for uniqueness in a (unanimity) bargaining model, where both the surplus to be divided and the order in which the players make proposals follow a general Markov process.

[3] See also Eraslan and Merlo (2002), Jackson and Moselle (2002).

as tax rates or minimum wages are chosen repeatedly over long time horizons, and thus may be subject to recurrent renegotiation. Bargaining when the status quo is not the worst outcome for all individuals is addressed in Banks and Duggan (2006) for general set-ups, and also considered in Cho and Duggan (2003). Baron (1996) addresses situations where one-dimensional policies are chosen repeatedly under simple majority rule, and decisions become the status quo for future negotiations. Bucovetsky (2003) discusses the effects of super-majority requirements in these environments.

Our results on the Pareto efficiency of super-majority rules are also a contribution to the literature on the endogenous emergence, efficiency and stability of majority rules. The general analysis of social choices over social choice rules is a classical problem. Its modern formalization starts with the discussion of the distinctive role of unanimous consent by Buchanan and Tullock (1962). Collective choices of majority rules under majoritarian regimes are discussed in Greenberg (1979), Caplin and Nalebuff (1988), and Barberà and Jackson (2004), among others. The reader is referred to the later for a discussion of this literature.

3. The model

A finite group of individuals $I \subset [0,1]$ must collectively select an alternative within the one-dimensional policy space $[0,1]$. There are $n \geqslant 3$ individuals in I, and n is assumed odd for expositional convenience. The interaction takes place over discrete time, $t = 0, 1, 2, \ldots$, and proceeds according to a stationary random proposers protocol given by a vector of recognition probabilities p, such that $\sum_{i \in I} p_i = 1$, $p_i > 0$ for all $i \in I$. At $t \geqslant 0$ individual i is selected to propose with recognition probability p_i. She then chooses an alternative in $[0,1]$ and all other players, sequentially in any fixed order, reply with acceptance or rejection. The proposal is approved if at least q players (including the proposer) accept it, where the required quota is a (super) majority, i.e. $q \in Q = \{(n+1)/2, \ldots, n\}$. Upon approval, the agreed alternative is implemented and the game ends. Otherwise, the game moves to $t+1$, a new proposer is selected, and so on.

Upon a collective decision that selects alternative $x \in [0,1]$ at date t, i obtains utility $\delta^t u(x,i)$, where $\delta \in (0,1)$ is a common discount rate. Utilities $u : [0,1]^2 \to \mathbb{R}_+$ are assumed to be continuous, concave and single-peaked in the first argument (thus, differentiable almost everywhere), and there are gains from any agreement; i.e. $u(i,i) > u(x,i) \geqslant 0$ for any $x \neq i$. Disagreement yields zero utility to all agents.

An individual (*pure*) *strategy* specifies actions – a proposal, and an acceptance/rejection rule – for each subgame. At a *stationary strategy* each individual makes the same proposal whenever she is selected and accepts/rejects proposals following a time independent rule. A *stationary subgame perfect equilibrium* (henceforth an *equilibrium*) is a profile of stationary strategies that are mutually best responses at each subgame.

We now turn to examine equilibrium strategies. We address their explicit characterization, and we show that they are unique.

4. The unique equilibrium

Our set-up is a particular case of the model examined in Banks and Duggan (2000). Their results assure that an equilibrium exists and that any equilibrium must be a no delay equilibrium in pure strategies. Hence, an equilibrium is fully described by the *approval set* – the subset of alternatives that get the acceptance of a q-majority – and each individual, whenever appointed to propose, proposes her most preferred alternative within this set. Furthermore, Lemma 1 of Cho and Duggan (2003) also applies immediately to our model: consequently the approval set is a non-empty compact interval. For expositional convenience we state these observations as Lemma 1.

Lemma 1. *The following hold*:

1. *(Banks and Duggan, 2000) An equilibrium exists, and it is a no delay equilibrium in pure strategies.*
2. *(Cho and Duggan, 2003) Consider an equilibrium. The set of proposals that receive approval is an interval* $[\underline{x}, \bar{x}]$, $0 \leqslant \underline{x} < \bar{x} \leqslant 1$.

Let us now turn attention to the conditions that determine the approval set in equilibrium. Consider an interval of alternatives $[\underline{x}, \bar{x}]$, and assume that it is the approval set of an equilibrium. Players must propose the alternative that they like best within this interval: That is, $x_i = \underline{x}$ for players $i < \underline{x}$, $x_i = \bar{x}$ for players $i > \bar{x}$, and players $i \in [\underline{x}, \bar{x}]$ must propose their peak. Congruently, the expected payoff (prior to appointment of the proposer) of a typical player i is

$$U_i[\underline{x}, \bar{x}] = P(\underline{x}) u(\underline{x}, i) + \sum_{j \in (\underline{x}, \bar{x}]} p_j u(j,i) + \bigl(1 - P(\bar{x})\bigr) u(\bar{x}, i),$$

where we write $P(x)$ to denote the cumulative probability of recognition to the left of x, i.e. $P(x) = \sum_{i \leqslant x} p_i$.

Thus, for each interval $[\underline{x}, \bar{x}]$ that is a possible approval set, we can compute the associated expected payoffs $U_i[\underline{x}, \bar{x}]$, and determine the corresponding *individual acceptance sets*. That is,[4]

[4] Note that the stationary assumption (with the assumption of continuity) implies that agents accept a proposal when indifferent.

$$A_i(\underline{x},\bar{x}) = \{z \in [0,1]:\ u(z,i) \geqslant \delta U_i[\underline{x},\bar{x}]\} = [\underline{x}_i, \bar{x}_i].$$

From the collection of individual acceptance sets, we must in turn construct the induced approval set. In equilibrium the latter must coincide with $[\underline{x},\bar{x}]$. Carrying this fixed point argument in full precision requires the intermediate observations and additional notation to which we turn our attention next.

Lemma 2 describes some general regularities of individual acceptance/rejection thresholds. We remark that the full strength of the assumption of concavity of the utilities is not necessary, the result relies only on single-peakedness and continuity of the utilities.

Lemma 2. *For an arbitrary interval $[x,z] \subset [0,1]$, let*

$$U_i[x,z] = P(x)u(x,i) + \sum_{j \in (x,z]} p_j u(j,i) + \big(1 - P(z)\big)u(z,i).$$

Consider $\bar{s}(x,i)$ and $\underline{s}(x,i)$ defined as

$$\bar{s}(x,i) = z \in (x,1] \quad \text{such that } u(z,i) = \delta U_i[x,z],$$

and

$$\underline{s}(x,i) = z \in [0,x) \quad \text{such that } u(z,i) = \delta U_i[z,x].$$

1. *If $\bar{s}(x,i)$ exists, then it is unique and continuous, and similarly for $\underline{s}(x,i)$.*
2. *There exist some threshold values $\underline{a}(i), \bar{a}(i), \underline{b}(i), \bar{b}(i) \in [0,1]$ such that $\bar{s}(x,i)$ exists if and only if $x \in [\underline{a}(i), \bar{a}(i)]$ and $\underline{s}(x,i)$ exists if and only if $x \in [\underline{b}(i), \bar{b}(i)]$.*

Now, by Lemma 2 we can define $\bar{\zeta} : [0,1]^2 \to [0,1]$, and $\underline{\zeta} : [0,1]^2 \to [0,1]$, as follows:

$$\bar{\zeta}(x,i) \equiv \begin{cases} \bar{s}(x,i), & \text{if } x \in [\underline{a}(i), \bar{a}(i)], \\ 1, & \text{otherwise;} \end{cases}$$

and

$$\underline{\zeta}(x,i) \equiv \begin{cases} \underline{s}(x,i), & \text{if } x \in [\underline{b}(i), \bar{b}(i)], \\ 0, & \text{otherwise.} \end{cases}$$

The approval set of an equilibrium must be $[\underline{x},\bar{x}] = [\underline{x}_i, \bar{x}_j]$, for some $i, j \in I$. Furthermore, by Lemma 2, the acceptance boundaries $\underline{x}_i$ and $\bar{x}_j$ must be a solution to

$$\underline{x}_i = \underline{\zeta}(\bar{x}_j, i) \quad \text{and} \quad \bar{x}_j = \bar{\zeta}(\underline{x}_i, j). \tag{1}$$

Our next result provides a sufficient condition assuring that Eq. (1) admits at most one solution.

Lemma 3. *Consider pair $(\underline{x},\bar{x})$ that solves Eq. (1) for $i, j \in I$. If $0 \leqslant j \leqslant i \leqslant 1$ and for every $y < j$ and $z > i$, the utilities satisfy*

$$\left|\frac{u_x(y,j)}{u_x(y,i)}\right| < 1 \quad \text{and} \quad \left|\frac{u_x(z,i)}{u_x(z,j)}\right| < 1 \quad \text{almost everywhere (a.e.)}, \tag{2}$$

then $(\underline{x},\bar{x})$ is unique.

Next, we turn attention to identifying individuals $i, j \in I$ that can be determinant of the approval set via Eq. (1). The following property of the utilities drastically simplifies this task.

Utility Symmetry (*SYM*): A group I satisfies *utility symmetry* if all $u(.,i)$ are identical up to a translation: $u(i+\varepsilon,i) = u(j+\varepsilon,j)$ for all $i, j \in I$ and any ε satisfying $i+\varepsilon, j+\varepsilon \in [0,1]$.

Lemma 4 establishes that under SYM the inequalities that determine the acceptance/rejection of the boundary policies $\underline{x}$ or $\bar{x}$ split the set I into two coalitions of contiguous individuals. We omit the dependence of U_i on $[\underline{x},\bar{x}]$ when no confusion arises.

Lemma 4. *Assume SYM. Fix an approval set $[\underline{x},\bar{x}]$. Then, the following hold*:

1. *If $u(\underline{x},i) \leqslant \delta U_i$ then $u(\underline{x},j) < \delta U_j$ for any j such that $j > i$.*
2. *If $u(\underline{x},i) \geqslant \delta U_i$ then $u(\underline{x},j) > \delta U_j$ for any j such that $j < i$.*

3. *If* $u(\bar{x}, i) \leqslant \delta U_i$ *then* $u(\bar{x}, j) < \delta U_j$ *for any* j *such that* $j < i$.
4. *If* $u(\bar{x}, i) \geqslant \delta U_i$ *then* $u(\bar{x}, j) > \delta U_j$ *for any* j *such that* $j > i$.

With Lemma 4 we are now ready to identify the crucial players.

Boundary players, l and r: The *right boundary player, r*, is the individual whose peak is located at the q-th position from the left, and the *left boundary player, l*, is the individual whose peak is at the q-th position from the right.

We establish next that when the group I satisfies SYM, in equilibrium the approval set must be the intersection of the acceptance sets of the boundary players l and r. This supplies a precise characterization of the equilibrium.

Proposition 1. *Assume SYM. Fix an equilibrium and let* $[\underline{x}, \bar{x}]$ *be the approval set. Then* $[\underline{x}, \bar{x}] = [\underline{x}_r, \bar{x}_l]$, *where*

$$\underline{x} = \underline{\zeta}(\bar{x}, r) \quad \text{and} \quad \bar{x} = \overline{\zeta}(\underline{x}, l). \tag{3}$$

Proof. It is immediate from Lemma 4 that:

1. If i rejects $\underline{x}$ then all $j > i$ also reject it (if i accepts $\underline{x}$ then all $j < i$ also accept it).
2. If i rejects $\bar{x}$ then all $j < i$ also reject it (if i accepts $\bar{x}$ then all $j > i$ also accept it).

Note that $\underline{x} = \underline{x}_r$ translates as

$$u(\underline{x}, r) = \delta U_r \quad \text{or} \quad \underline{x} = 0 \quad \text{if } u(0, r) \geqslant \delta U_r.$$

We show by contradiction that this condition is necessary. Suppose that $\underline{x} > \underline{x}_r$. This implies $u(\underline{x}, r) > \delta U_r$ and therefore, by Lemma 4, $u(\underline{x}, i) > \delta U_i$ for all $i < r$. By continuity and single-peakedness, there exists $y \in (\underline{x}_r, \underline{x})$ such that $u(y, i) > \delta U_i$ for all $i \leqslant r$. Thus, $y < \underline{x}$ would be accepted by q players, contradicting that $\underline{x}$ is the lower bound of the approval set. Similarly, if $\underline{x} < \underline{x}_r$ then $u(\underline{x}, r) < \delta U_r$ then (again by Lemma 4) more than $n - q$ of players would reject $\underline{x}$, contradicting that $\underline{x}$ lies in the approval set. A similar argument applies to show that $\bar{x} = \bar{x}_l$ is necessary.

It is immediate that the condition is also sufficient, since when $[\underline{x}, \bar{x}] = [\underline{x}_r, \bar{x}_l]$ all proposals in $[\underline{x}, \bar{x}]$ are accepted by q players, and all that lie outside are rejected by $n - q + 1$ players.

Hence, we have established that an interval $[\underline{x}, \bar{x}]$ is the approval set of an equilibrium if and only if it is the intersection of the acceptance sets of the two boundary players l and r. Now, by Lemma 2 the values of $\underline{x}$ and $\bar{x}$ must be a fixed point of the function $\overline{\zeta}(., l), \underline{\zeta}(., r) : [0, 1]^2 \to [0, 1]^2$, which yields Eq. (3). □

We are now ready to state our main result, Proposition 2, that establishes that SYM is also a sufficient condition for the uniqueness of the equilibrium.[5]

Proposition 2 *(Uniqueness of the equilibrium). Assume SYM. Then there is a unique equilibrium.*

Proof. By Proposition 1, $\underline{x}$ and $\bar{x}$ must solve Eq. (3). It is immediate to check that under SYM the conditions of Lemma 3 hold for $i = r$ and $j = l$. Hence (3) admits only one solution. Therefore, the uniqueness of the approval set (and the uniqueness of the equilibrium) follows. □

Eq. (3) supplies the tool for the explicit computation of the approval set. Fig. 1 displays its determination for a group of $n = 11$ satisfying EQP and URP, $q = 7, u(x, i) = 1 - (x - i)^2$ and $\delta = 0.99$.

More importantly, Proposition 2 opens the door to comparative statics exercises to measure the effects of changes in q. We turn attention to this next.

5. Comparative statics and Pareto optimal rules

Thanks to the uniqueness of the equilibrium and the explicit computation of approval sets, we can examine the variation of equilibrium outcomes across different majority requirements. In order to simplify notation, in this section we assume that all agents have different peaks.[6] The following definitions will also be used in the sequel:

[5] Although SYM is not necessary for uniqueness, multiple equilibria are obtained when no restrictions on the preferences of the players are imposed (see Example 1 in Cho and Duggan, 2003).

[6] This assumption only implies that the boundary players vary with the quota. Otherwise, it might be possible that a change in the quota does not affect the equilibrium acceptance set so that comparative statics would be trivial.

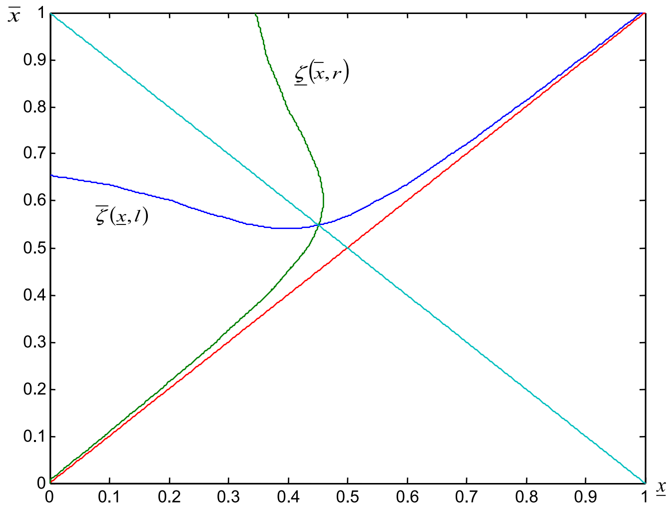

Fig. 1.

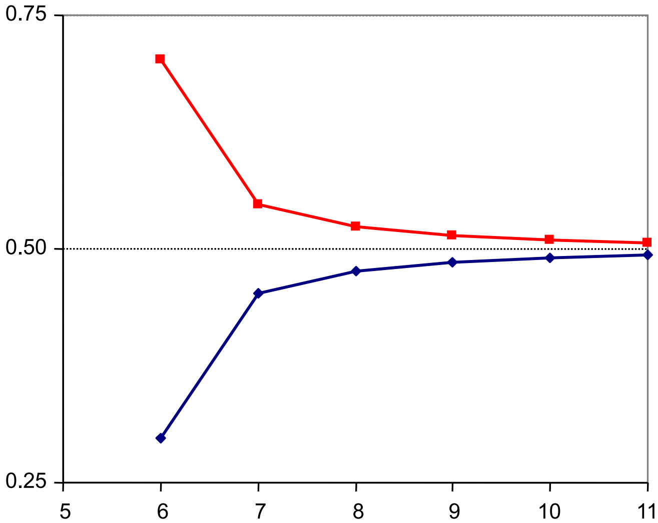

Fig. 2. The approval sets under EQP and URP as a function of q.

Strong Utility Symmetry (*SSYM*): The group I satisfies SYM and utilities $u(.,i)$ are *symmetric around the peak*; that is, there is a decreasing, concave and continuous function v such that $u(x,i) = v(|i-x|)$.

Peak Location Symmetry (*PLS*): A group of individuals I satisfies *peak location symmetry* if for every individual $i \in I$ there is an individual $j = 1 - i \in I$.

Equidistant Peaks (*EQP*): A group of individuals I satisfies *equidistant peaks* if $I = \{0, 1/(n-1), 2/(n-1), \ldots, 1\}$.

Uniform Recognition Probabilities (*URP*): A vector of recognition probabilities p satisfies *uniform recognition probabilities* when $p_i = 1/n$ for all $i \in I$.

Weak Recognition Symmetry (*WRS*): A vector of recognition probabilities p satisfies *weak recognition symmetry* when $\sum_{i\in I,\, i<1/2} p_i = \sum_{i\in I,\, i>1/2} p_i$.

As an illustration, Figs. 2 to 4 display the approval sets as a function of q for some examples with $n = 11$ individuals, $u(x,i) = 1 - (i-x)^2$, and $\delta = 0.99$.

When a unique equilibrium prevails for each q, we may examine efficiency properties of the different $q \in Q$. Write the approval set under majority rule q as $[\underline{x}(q), \bar{x}(q)]$. Now, the preferences of individual i over $q \in Q$ are naturally given by $U_i(q) \equiv U_i[\underline{x}(q), \bar{x}(q)]$. The minimal requirement of efficiency is Pareto optimality:

Pareto optimal q. We will say that $q \in Q$ is *a Pareto optimal majority rule* for (I, u, p, δ) if there is no $q' \in Q$, $q' \neq q$ such that $U_i(q') \geqslant U_i(q)$ for all $i \in I$, with at least one strict inequality.

We first observe that the unanimity rule is always Pareto optimal.

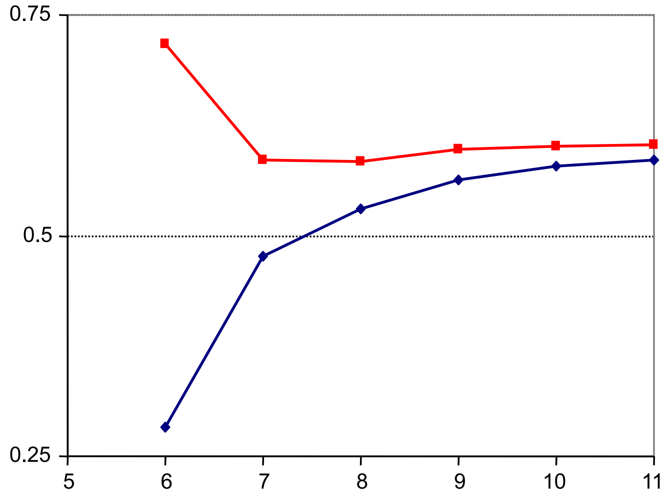

Fig. 3. The approval sets under EQP and $p = (1/41)(6, 6, 5, 5, 4, 4, 3, 3, 2, 2, 1)$ as a function of q.

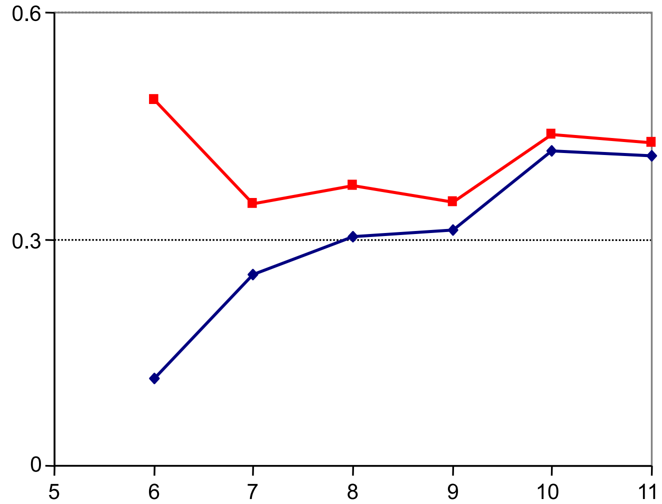

Fig. 4. The approval sets under URP and $I = \{0, 0.1, 0.11, 0.2, 0.21, 0.3, 0.4, 0.5, 0.6, 0.9, 1\}$ as a function of q.

Proposition 3. *Assume SYM. Unanimity is always Pareto optimal.*

Next, we explore environments for which we can examine the full set of Pareto optimal rules. Consider the equilibrium outcomes displayed in Fig. 2, and note that the approval sets shrink symmetrically around $x = 1/2$ while the expected outcome remains unchanged as q increases. For strictly concave utilities, this implies that individual expected utilities strictly increase in q. Hence, stronger super-majorities Pareto dominate weaker ones. We argue next that the features displayed in this example are rather general. Proposition 4 establishes that under SSYM, PLS and URP the approval set shrinks symmetrically around $x = 1/2$ as q increases.

Proposition 4. (*Decreasing q induces a mean preserving spread of the approval set.*) *Assume SSYM, PLS and URP. If $q < q'$, then $\underline{x}(q) < \underline{x}(q') < \bar{x}(q') < \bar{x}(q)$. Moreover, the probability distribution of equilibrium outcomes under q is a mean preserving spread of the probability distribution of equilibrium outcomes under q'.*

It is well known that for any strictly concave utility a mean preserving spread induces a decrease in expected utility. Hence, when the preferences are strictly concave all $i \in I$ strictly prefer higher quotas. When the preferences are not strictly concave, expected utilities still increase in q for players that have their peak in $[\underline{x}(n), \bar{x}(n)]$,[7] while for other players a mean preserving spread may leave them indifferent. Hence, the following holds.

[7] Note that there is always at least one agent $i \in [\underline{x}(n), \bar{x}(n)]$.

Proposition 5. *Assume SSYM, PLS and URP. The unanimity rule, $q = n$, is the unique Pareto optimal rule.*

When PLS or URP fails, Pareto optimality no longer selects unanimity uniquely and many (or all) $q \in Q$ are Pareto optimal. This is easy to see in Fig. 3, which displays the approval sets for set-ups where URP fails. Note that as q increases the size of the approval set shrinks, and the average outcome drifts so that a decrease in q leads to a spread which is not mean preserving. Hence, some agents are better off at weak majorities while others prefer stronger super-majorities.

However, URP is not a necessary condition for the unique Pareto optimality of the unanimity rule. Under the assumption of sufficient patience, weaker sufficient conditions are given in the following proposition.

Proposition 6. *Assume SSYM, PLS and WRS. There exists $\bar{\delta} < 1$ such that for all $\delta \in [\bar{\delta}, 1)$ unanimity Pareto dominates all other $q \in Q$.*

Next, we display 2 examples where PLS holds but WRS fails.[8]

An example where all $q \in Q$ are Pareto optimal. Let $I = \{0, 1/4, 1/2, 3/4, 1\}$, $u(x; i) = 1 - (x - i)^2$, $p = (1/9, 1/3, 1/3, 1/9, 1/9)$ and $\delta = 0.995$. The table displays approval sets and expected utilities for each $q = 3, 4, 5$. It is immediate to see that all q are Pareto optimal.

q	3	4	5
$\underline{x}$	0.378 13	0.48514	0.49385
$\bar{x}$	0.621 87	0.50382	0.50134
U_0	0.76718	> 0.75565	> 0.75242
$U_{1/4}$	0.94114	> 0.94028	> 0.9387
$U_{1/2}$	0.9901	< 0.9999	< 0.99998
$U_{3/4}$	0.91406	< 0.93452	< 0.93627
U_1	0.71302	< 0.74414	< 0.74755

An example where Pareto optimal rules are a strict subset of Q. Take I, u, δ as in the previous example and let $p = (1/6, 1/3, 1/6, 1/6, 1/6)$. Now, $q = 4, 5$ are Pareto optimal but $q = 3$ is not.

q	3	4	5
$\underline{x}$	0.32892	0.48317	0.49276
$\bar{x}$	0.67108	0.50191	0.50026
U_0	0.75412	< 0.75764	> 0.75351
$U_{1/4}$	0.92737	< 0.94125	> 0.93924
$U_{1/2}$	0.97561	< 0.99986	< 0.99997
$U_{3/4}$	0.89885	< 0.93347	< 0.93571
U_1	0.6971	< 0.74208	< 0.74644

6. Final remarks

We have assumed general locations of peaks and (stationary) random recognition probabilities, under voting rules that assign uniform voting rights to all players. Our uniqueness result generalizes easily to a set-up where the assignment of voting rights is not uniform: one would simply need to redefine the boundary players to account for asymmetries in voting rights.

Assuming SYM – that all agents display the same utilities up to a translation – has a crucial role in our uniqueness result. However SYM is not necessary; it is only a very tractable property that delivers what is really crucial to assure the uniqueness of the equilibrium: (i) the boundary players always uniquely determine the equilibrium approval set, and (ii) the utilities of these players satisfy a regularity property. Moreover, SYM also limits the importance of the recognition probabilities on the equilibrium outcome.[9] Since the boundary players l and r remain determinant irrespective of the recognition probabilities, as players become patient the outcome must always be near to $[l, r]$. In particular when q is the simple majority, the unique boundary player $l = r$ is the median voter. Then, an obvious but important implication is that as impatience vanishes, the median peak prevails as the unique equilibrium outcome regardless of the recognition probability distribution or the location of the other peaks.[10]

An important feature of our model is that an immediate agreement prevails in equilibrium regardless of the majority required. In real negotiations, disagreements are a real possibility, and the chances that a proposal fails to get approval seems greater the greater the majority needed. This suggests that in bargaining processes where frictions yield inefficient outcomes, the forces favoring super-majorities are more limited.

[8] Similar conclusions would be obtained by considering URP and asymmetric locations of the peaks (see Fig. 4).

[9] The importance of proposal rights over voting rights and impatience is stressed by Kalandrakis (2006).

[10] Cho and Duggan (2009) prove such a claim in a more general setting where this convergence is obtained. In particular they allow for non-stationary strategies and (possibly) history-dependent recognition probabilities.

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Appendix A

Proof of Lemma 2. (1) Fix $x \in [0,1]$ and define $\kappa(z;x) = u(z,i) - \delta U_i[x,z]$, so that $\kappa'(z;x) = u_x(z,i)(1-\delta(1-P(z)))$ almost everywhere (a.e.) is positive when $z < i$, and negative at $z > i$. Hence, by single-peakedness $\kappa(z;x) = 0$ has at most two solutions: one with $z \leqslant i$ and another with $z > i$. Notice, however, that $\kappa(x;x) = u(x,i)(1-\delta) > 0$ so that at most one solution $z > x$ exists. Moreover, as $\kappa(z;x)$ is continuous the solution must be.

(2) Next, we show that when $\bar{s}(x,i)$ and $\bar{s}(y,i)$ do exist, with $x \leqslant y$, then $\bar{s}(w,i)$ exists for any $w \in [x,y]$.

Assume otherwise, i.e., $\kappa(z;w) = u(z,i) - \delta U_i[w,z] > 0$ for all $z \geqslant i$. Then, in particular

$$\kappa(1;w) = u(1,i) - \delta U_i[w,1] > 0, \tag{4}$$

$$\kappa(1;x) = u(1,i) - \delta U_i[x,1] \leqslant 0, \tag{5}$$

$$\kappa(1;y) = u(1,i) - \delta U_i[y,1] \leqslant 0. \tag{6}$$

We distinguish two cases: either (a) $u(w,i) \geqslant u(y,i)$ or (b) $u(x,i) < u(w,i) < u(y,i)$.

In case (a), by single-peakedness $u(z,i) \geqslant u(y,i)$ for all $z \in (w,y)$. Thus,

$$\begin{aligned} U_i[w,1] &= P(w)u(w,i) + \sum_{j\in(w,1]} p_j u(j,i) \geqslant P(w)u(y,i) + \sum_{j\in(w,y]} p_j u(j,i) + \sum_{j\in(y,1]} p_j u(j,i) \\ &\geqslant P(w)u(y,i) + \big[P(y) - P(w)\big]u(y,i) + \sum_{j\in(y,1]} p_j u(j,i) \\ &= P(y)u(y,i) + \sum_{j\in(y,1]} p_j u(j,i) = U_i[y,1]. \end{aligned}$$

Now, using (4) and (6) we obtain a contradiction.

In case (b) we have that $u(z,i) \leqslant u(w,i)$ for all $z \in (x,w)$. Thus,

$$\begin{aligned} U_i[x,1] &= P(x)u(x,i) + \sum_{j\in(x,1]} p_j u(j,i) = P(x)u(x,i) + \sum_{j\in(x,w]} p_j u(j,i) + \sum_{j\in(w,1]} p_j u(j,i) \\ &\leqslant P(x)u(w,i) + \big[P(w) - P(x)\big]u(w,i) + \sum_{j\in(w,1]} p_j u(j,i) \\ &= P(w)u(w,i) + \sum_{j\in(w,1]} p_j u(j,i) = U_i[w,1], \end{aligned}$$

so that (4) and (5) cannot be simultaneously satisfied. □

Proof of Lemma 3. Direct computations establish that[11]

$$\bar{\zeta}_x(\underline{x},j) = \frac{\delta P(\underline{x})}{1-\delta+\delta P(\bar{x})} \frac{u_x(\underline{x},j)}{u_x(\bar{x},j)} \quad \text{for } \underline{x} \in \big(\underline{a}(j), \bar{a}(j)\big) \text{ a.e.}$$

where, by construction $\bar{x} = \bar{\zeta}(\underline{x},j) > \max\{\underline{x},j\}$.

Similarly,

$$\underline{\zeta}_x(\bar{x},i) = \frac{\delta(1-P(\bar{x}))}{1-\delta P(\underline{x})} \frac{u_x(\bar{x},i)}{u_x(\underline{x},i)} \quad \text{for } \bar{x} \in \big(\underline{b}(r), \bar{b}(r)\big) \text{ a.e.}$$

where, by construction $\underline{x} = \underline{\zeta}(\bar{x},i) < \min\{\bar{x},i\}$.

Let us define $H(x) = \bar{\zeta}(\underline{\zeta}(x,i),j)$. Thus, for any $\bar{x} \in (\underline{b}(r), \bar{b}(r))$, the following holds a.e.,

[11] Note that the existence of $\bar{s}(\underline{x},j)$ implies $u_x(\bar{x},j) < 0$ and the existence of $\underline{s}(\bar{x},i)$ implies $u_x(\underline{x},i) > 0$ when such derivatives do exist.

$$H'(\bar{x}) = \frac{\delta P(\underline{x})}{1-\delta+\delta P(\bar{x})} \frac{\delta(1-P(\bar{x}))}{1-\delta P(\underline{x})} \frac{u_x(\underline{x}, j)}{u_x(\bar{x}, j)} \frac{u_x(\bar{x}, i)}{u_x(\underline{x}, i)}$$
$$= A(\underline{x}, \bar{x}) \frac{u_x(\underline{x}, j)}{u_x(\bar{x}, j)} \frac{u_x(\bar{x}, i)}{u_x(\underline{x}, i)} \quad \text{with } 0 < A(\underline{x}, \bar{x}) < 1.$$

By construction, $H(\bar{x}) = \bar{\zeta}(0, j)$ for all $\bar{x} \notin (\underline{b}(i), \bar{b}(i))$. For any $\bar{x} \in (\underline{b}(i), \bar{b}(i))$ we distinguish two cases: either (a) $\bar{x} \leqslant i$ or (b) $\bar{x} > i$. In case (a) either $\underline{x} < j$, in which case we obtain that $H'(\bar{x}) < 0$ a.e., or $\underline{x} \geqslant j$ and therefore, by concavity

$$0 \leqslant \frac{u_x(\underline{x}, j)}{u_x(\bar{x}, j)} \leqslant 1 \quad \text{and} \quad 0 \leqslant \frac{u_x(\bar{x}, i)}{u_x(\underline{x}, i)} \leqslant 1 \quad \text{a.e.,}$$

so that $0 \leqslant H'(\bar{x}) < 1$ a.e. In case (b) we obtain that $H'(\bar{x}) < 0$ a.e. if $\underline{x} > j$, and when $\underline{x} \leqslant j$, by (2), we obtain that $|H'(\bar{x})| < 1$ a.e.

Hence, as $H(\bar{x})$ is continuous and $H'(\bar{x}) < 1$ a.e., there is at most one $\bar{x}$ solving $H(\bar{x}) = \bar{x}$, and hence at most one pair $(\underline{x}, \bar{x})$ solving Eq. (1). □

Proof of Lemma 4. We prove statement 1. A similar argument applies to prove 2, 3 and 4.

Assume otherwise, i.e., $u(\underline{x}, j) \geqslant \delta U_j$ for some $j > i$ such that $u(\underline{x}, i) \leqslant \delta U_i$. Hence,

$$u(\underline{x}, i) \leqslant \delta\left(P(\underline{x})u(\underline{x}, i) + \sum_{h\in(\underline{x},\bar{x}]} p_h u(h, i) + \big(1 - P(\bar{x})\big)u(\bar{x}, i)\right) \tag{7}$$

and

$$u(\underline{x}, j) \geqslant \delta\left(P(\underline{x})u(\underline{x}, j) + \sum_{h\in(\underline{x},\bar{x}]} p_h u(h, j) + \big(1 - P(\bar{x})\big)u(\bar{x}, j)\right). \tag{8}$$

Let $d_{ij} = j - i$ denote the distance between the peaks of the players i and j. By SYM $u(x, j) = u(x - d_{ij}, i)$ for all $x - d_{ij} \geqslant 0$ and $u(x, i) = u(x + d_{ij}, j)$ if $x + d_{ij} \leqslant 1$. Moreover, by concavity it must be that $u(x, i) - u(x, j)$ is decreasing in x. To see that, note that for $x \leqslant 1 - d_{ij}$ we have that $u(x, i) - u(x, j) = u(x + d_{ij}, j) - u(x, j)$, which by concavity is decreasing. Likewise, for $x \geqslant d_{ij}$ we have that $u(x, i) - u(x, j) = u(x, i) - u(x - d_{ij}, i)$, which by concavity is decreasing. Note also that if $1 - d_{ij} < d_{ij}$ (thus, $d_{ij} > 1/2$) it must be that $u(x, i)$ is decreasing and $u(x, j)$ is increasing in the interval $(1 - d_{ij}, d_{ij})$. Otherwise, either $i > 1 - d_{ij}$ implies $j - i \leqslant 1 - i < d_{ij}$ or $j < d_{ij}$ implies $j - i \leqslant j < d_{ij}$. Thus, $u(x, i) - u(x, j)$ is also decreasing for $x \in (1 - d_{ij}, d_{ij})$.

Adding up Eqs. (7) and (8) we have that $u(\underline{x}, i) - u(\underline{x}, j)$ is not greater than

$$\delta\left(P(\underline{x})\big(u(\underline{x}, i) - u(\underline{x}, j)\big) + \sum_{h\in(\underline{x},\bar{x}]} p_h\big(u(h, i) - u(h, j)\big) + \big(1 - P(\bar{x})\big)\big(u(\bar{x}, i) - u(\bar{x}, j)\big)\right).$$

However, as $u(x, i) - u(x, j)$ is decreasing we obtain that the above expression is smaller than $\delta(u(\underline{x}, i) - u(\underline{x}, j))$, which is a contradiction. □

Proof of Proposition 3. Note that $\bar{\zeta}(\underline{x}, 0)$ and $\underline{\zeta}(\bar{x}, 1)$ are increasing functions respectively in $\underline{x}$ and $\bar{x}$. We first show that for any $q < n$, either (a) $\underline{x}(n) > \underline{x}(q)$ and $\bar{x}(n) < \bar{x}(q)$; (b) $\underline{x}(n) < \underline{x}(q)$ and $\bar{x}(n) < \bar{x}(q)$ or (c) $\underline{x}(n) > \underline{x}(q)$ and $\bar{x}(n) > \bar{x}(q)$. Assume otherwise. I.e., $\underline{x}(n) < \underline{x}(q)$ and $\bar{x}(n) > \bar{x}(q)$. This implies that $\bar{\zeta}(\underline{x}(q), 0) \geqslant \bar{\zeta}(\underline{x}(n), 0) = \bar{x}(n) > \bar{x}(q)$. Thus, $u(\bar{x}(q), 0) > \delta U_0[\underline{x}(q), \bar{x}(q)]$, which by Lemma 4(4) implies $u(\bar{x}(q), l) > \delta U_l[\underline{x}(q), \bar{x}(q)]$, which is a contradiction.

In case (c) clearly agent 1 is worse off, while in case (b) agent 0 gets a lower expected utility by reducing unanimous consent. In case (a) the variance increases and the mean may either (i) increase, in which case agent 0 is worse; (ii) decrease, thus worsening player 1 or (iii) remain constant, implying a mean preserving spread so that all agents are worse off.

That is, it is impossible to reduce the quota and increase the expected utility of all agents so that $q = n$ is Pareto optimal. □

Lemma 5. *Assume SYM. $\bar{\zeta}(\underline{x}, l)$ is increasing in l, and $\underline{\zeta}(\bar{x}, r)$ is increasing in r.*

Proof. Fix $\underline{x}$ and consider $\bar{\zeta}(\underline{x}, l)$ at two possible values of $l = a, b$ with $a < b$. Notice that $\bar{\zeta}(\underline{x}, a) < 1$ implies $u(\bar{\zeta}(\underline{x}, a), a) = \delta U_a$. In this case, if $\bar{\zeta}(\underline{x}, b) = 1$ the result follows. Otherwise, $\bar{\zeta}(\underline{x}, b)$ is defined by $u(\bar{\zeta}(\underline{x}, b), b) = \delta U_b$. Moreover, by Lemma 4(3), $u(\bar{\zeta}(\underline{x}, b), a) < \delta U_a$. Thus, $u(\bar{\zeta}(\underline{x}, b), a) < u(\bar{\zeta}(\underline{x}, a), a)$. Moreover, as $\bar{\zeta}(\underline{x}, b) > b > a$ and $\bar{\zeta}(\underline{x}, a) > a$ we have that $\bar{\zeta}(\underline{x}, a) < \bar{\zeta}(\underline{x}, b)$. In case that $\bar{x} = \bar{\zeta}(\underline{x}, a) = 1$, it must be that $u(1, a) \geqslant \delta U_a$. Therefore, by Lemma 4(4), $u(1, b) > \delta U_b$ so that $\bar{\zeta}(\underline{x}, b) = 1$. □

Proof of Proposition 6. Step 1: Fix $[\underline{y}, \overline{y}] = [1/2 - \varepsilon, 1/2 + \varepsilon]$ where ε is the minimal distance between agent 1/2 and any other player. We next show that there is $\overline{\delta} < 1$ where $[\underline{x}, \overline{x}] \subset [\underline{y}, \overline{y}]$ is the equilibrium approval set when $q = (n+1)/2$. To be an equilibrium approval set, $\underline{x} = 1 - \overline{x}$ must satisfy

$$u(\underline{x}, 1/2) = \delta\big(\big(P(1/2) - p_{1/2}\big)u(\underline{x}, 1/2) + p_{1/2} + \big(1 - P(1/2)\big)u(\underline{x}, 1/2)\big).$$

I.e.,

$$u(\underline{x}, 1/2)\big[1 - \delta[1 - p_{1/2}]\big] = \delta p_{1/2}.$$

Thus, for any

$$\delta \geqslant \overline{\delta} = \frac{u(\underline{y}, 1/2)}{u(\underline{y}, 1/2)(1 - p_{1/2}) + p_{1/2}} \in (0, 1),$$

we obtain that $[\underline{x}, \overline{x}] \subset [\underline{y}, \overline{y}]$ is the equilibrium approval set.

Step 2: We next show that if $\delta \geqslant \overline{\delta}$ then increasing the majority requirement reduces the approval set symmetrically around 1/2.

Let l and r denote the (symmetric) boundary players. Let $\underline{w}(\bar{z}, r)$ and $\overline{w}(\underline{z}, l)$ solve

$$u(w, r) - \delta\big(\big[P(1/2) - p_{1/2}\big]u(w, r) + p_{1/2}u(1/2, r) + \big[1 - P(1/2)\big]u(\bar{z}, r)\big) \equiv 0$$

and

$$u(w, l) - \delta\big(\big[P(1/2) - p_{1/2}\big]u(\bar{z}, l) + p_{1/2}u(1/2, l) + \big[1 - P(1/2)\big]u(w, l)\big) \equiv 0$$

respectively.

By PLS ($l = 1 - r$) the solution to

$$\underline{z} = \underline{w}(\bar{z}, r),$$
$$\bar{z} = \overline{w}(\underline{z}, l)$$

must be symmetric. Moreover, it can be shown that $\underline{w}(\bar{z}, r) < \underline{w}(\bar{z}, r')$ for any $r' > r$ and $\overline{w}(\underline{z}, l) > \overline{w}(\underline{z}, l')$ for any $l < l'$. Therefore, as r increases and l decreases with q, we have that $\underline{w}(\bar{z}, r) = \underline{\zeta}(\bar{z}, r)$ and $\overline{w}(\underline{z}, l) = \overline{\zeta}(\underline{z}, l)$ for any $\delta \geqslant \overline{\delta}$, and the result follows. □

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