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Bargaining over multiple issues with maximin and leximin preferences

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Abstract Global bargaining problems over a finite number of different issues are formalized as cartesian products of classical bargaining problems. For maximin and leximin bargainers we characterize global bargaining solutions that are efficient and satisfy the requirement that bargaining separately or globally leads to equivalent outcomes. Global solutions in this class are constructed from the family of monotone path solutions for classical bargaining problems.

1 Introduction

We address bargaining problems where agreements must resolve several different issues. The following examples illustrate what situations concern us:

1. A group must share a basket of ingredients to produce paella; each individual follows his own recipe requiring to mix ingredients in exactly the right proportions, and all wish to cook as much as possible.
2. Bargaining at the World Trade Organization, when each representative is an egalitarian government bargaining on behalf of her country. Within each country, each issue affects disjoint subsets of the population, and transfers between groups are impossible.

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3. Bargaining under uncertainty without expected utility. Ex-ante, agents must reach a contingent agreement for each state of the world.

The literature addressing multiple issue bargaining has focussed mostly on environments where the agents' preferences over benefit bundles are given by a utility function; then a global bargaining problem can be reduced to a classical bargaining problem where feasible agreements are allocations of utilities for each player. In this case, addressing the global problem is tantamount to uncovering the structure of the feasible set of utilities from that of single issue bargaining problems, and to discovering the links between the classical solutions applied separately to the issues and classical solutions applied to the set of global feasible utilities. This approach is usually pursued under the additional assumption that utilities are additive across issues.[1]

In contrast, we wish to consider solutions that apply directly to the global bargaining situation, either because detailed information about preferences is limited or manipulable; or because the preferences of the bargainers do not admit a utility representation that allows bargaining on utility allocations.

Bossert et al. (1996) and Bossert and Peters (2001), that stress the motivation of bargaining under uncertainty, precede us in addressing global bargaining problems as cartesian products of classical bargaining problems.[2] In Bossert et al. (1996) a class of strictly monotone path solutions is characterized by imposing maximin ex-ante efficiency. Bossert and Peters (2001) consider efficient solutions when agents have minimax regret preferences; in bilateral problems the class of monotone utopia-path solutions is characterized, but for more agents only dictatorial solutions remain. Bossert and Peters (2002) consider uncertainty with respect to the disagreement point and characterize a wider class of monotone path solutions, also by imposing a condition of maximin efficiency.

Hinojosa et al. (2005) and Mármol et al. (2007) discuss solutions exhibiting a maximin efficiency propriety for multi-criteria bargaining models. The main difference with our approach is that their bargaining sets do not admit a representation as a cartesian product of separate problems.

Our analysis begins with focus on environments with minimax preferences. We point out that the global solutions that assure efficient outcomes have a direct characterization in terms of efficiency on the intersection of the issue bargaining sets. Furthermore, we show that the requirement of equivalence in "minimax result", regardless of whether the issues are addressed separately or globally, characterizes the family of monotonic solutions.

Our main results relate to environments where preferences are leximin. We characterize the set of efficient outcomes, and we propose a family of global solutions that attains efficient outcomes. This family is constructed by applying a classical solution to a sequence of classical bargaining problems: First the

1 See Kalai (1977), Myerson (1981), Binmore (1984), Gupta (1989) and Ponsatí and Watson (1997).

2 See also Bossert and Peters (2000).

intersection of all the issue problems is considered, and (at least) one issue is resolved; subsequently the disagreement point and the bargaining set to address next is revised, and so on until a global agreement is obtained. We show that to attain the same outcome whether the issues are addressed separately or globally via this step by step procedure it is necessary and sufficient to use a monotonic classical solution.

The link between bargaining solutions and sequential procedures was previously discussed by Ponsatí and Watson (1997) and O'Neill et al. (2004). A main concern in these papers is the characterization of solutions delivering outcomes that are independent of the agendas, assumed exogenous. In contrast, in the family of solutions that we propose agendas arise endogenously.

The rest of the paper is organized as follows. We lay out the set up and discuss preliminary observations in Sect. 2. Solutions for global bargaining under maximin preferences are addressed in Sect. 3. Sections 4 and 5 address environments with leximin bargainers.

2 Multiple issue bargaining

The following notation is used: $x, y \in \Re^n$: $x > y$ means that $x_j > y_j$, for $j = 1, \ldots, n$; $x \geq y$ means that $x_j \geq y_j$, for $j = 1, \ldots, n$ and $x \neq y$; and $x \geqq y$ means that $x_j \geq y_j$, for $j = 1, \ldots, n$. For a matrix $X \in \Re^{m \times n}$ we denote by X_j the jth row, by X^i the ith column, and by X^t its transpose. Dominance relations between matrices $X, Y \in \Re^{m \times n}$, are denoted as follows: $X \geqq Y$ if $x^i_j \geq y^i_j$, $\forall\, i, j$; $X \geq Y$ if $X \geqq Y$ and $X \neq Y$; $X > Y$ if $x^i_j > y^i_j$, $\forall\, i, j$.

A group of n agents, $i = 1, 2, \ldots, n$, bargain over m different issues, $j = 1, \ldots, m$. $N = \{1, \ldots, n\}$ is the set of agents, and $M = \{1, \ldots, m\}$ is the set of issues. The *issue bargaining problems* are classical bargaining problems; they are represented by pairs (S_j, d_j), where $S_j \subset \Re^n$ is the set of feasible benefits that can be allocated to the agents by agreement on issue j and $d_j \in \Re^n$ is the allocation of benefits from disagreement on that issue.

The sets S_j are compact and strictly d-comprehensive, that is, when $x \in S_j$, if $d \leq y \leq x$, then $y \in S_j$, and $z \in S_j$ exists such that $z > y$. Comprehensiveness suffices for many results, but we keep the stronger assumption throughout the paper for expositional simplicity. We also assume that $s > d_j$ for some $s \in S_j$. The set of efficient allocations is denoted as $e(S_j)$, $e(S_j) = \{s \in S_j, \nexists s' \in S_j, s' \geq s\}$, the set of weakly efficient allocations by $we(S_j)$, $we(S_j) = \{s \in S_j, \nexists s' \in S_j, s' > s\}$, and the class of classical bargaining problems by $\mathcal{B}$. We also denote by $\mathcal{B}_0$ the subclass of $\mathcal{B}$ with disagreement point at the origin. A *classical solution* is a function $\varphi : \mathcal{B} \to \Re^n$ such that $\varphi(S_j) \in e(S_j)$. We further assume that agents can measure the benefit from each issue, and compare benefits across issues.

A *global bargaining problem* is a pair (S, d), $S = S_1 \times S_2 \times \ldots \times S_m$ is the set of feasible outcomes, each allocating feasible benefits on every issue, while $d \in \Re^{m \times n}$ is the *status quo*, the allocation of benefits under disagreement over all issues. Thus, a *global* agreement is an element of S, an $m \times n$ matrix:

$$X = \begin{pmatrix} x_1^1 & x_1^2 & \dots & x_1^n \\ x_2^1 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \dots & x_m^n \end{pmatrix}$$

The ith column of X, $X^i = (x_1^i, x_2^i, \dots, x_m^i)^t \in \Re^m$, represents the benefits for player i, in each of the m issues. The jth row of X, $X_j = (x_j^1, x_j^2, \dots, x_j^n) \in S_j \subset \Re^n$ is the allocation of issue j benefits among the players.

Let $\mathcal{GB}$ denote the class of global bargaining games. We wish to explore *solutions*, criteria to select outcomes for each $(S, d) \in \mathcal{GB}$, and discuss their properties and performance. Formally a *solution* is a correspondence, $F : \mathcal{GB} \to \Re^{m\times n}$ such that $F(S) \subseteq S$, that selects a non-empty subset of feasible outcomes (possibly a singleton) for each global bargaining problem. For simplicity we fix the disagreement point at the origin and let $S_j \subset \Re^n_+$, so that global bargaining problems are represented by $S = \prod_{j\in M} S_j$. Denote the class of such bargaining games by $\mathcal{GB}_0$.

A global problem can be thought as m separate bargaining problems. Then a classical solution applied to each issue yields a solution to the global problem. However, in global bargaining players can give up benefits in one issue for gains in another, while such trade-offs are precluded when issues are discussed separately. The global problem may also be approached as a bargaining problem among $n \times m$ agents; yet global payoffs - attained by the combination benefits over the m-issues—remain ignored.[3] It is, therefore, worthwhile to explore global solutions that genuinely address issues jointly taking into account the interrelations in the benefits of different issues over the total payoffs of the agents, considering properties that seem appropriate to resolve global problems, and exploring their links to classical solutions.

2.1 Pareto optimality

A precise notion of efficiency for global problems requires a specification of the agents' preferences over bundles of issue benefits.

Prior to examining more detailed specifications for the individual preferences, let us consider the sets of outcomes that are undominated in the natural dominance relations $X \geq Y$ and $X > Y$. That is, the dominance relations based only on the assumption that individual preferences are monotone. We will refer to such undominated outcomes as Pareto optimal.

Pareto Optimality (PO): $X \in S \subseteq \Re^{m\times n}$ is *Pareto optimal* in S, if $\nexists Y \in S$, such that $Y \geq X$.

[3] See Bergstresser and Yu (1977).

Since $S = S_1 \times \cdots \times S_m$, a global solution is Pareto optimal if and only if it allocates the benefits of each issue efficiently. Thus the following holds.

Proposition 1 *For $S = S_1 \times \cdots \times S_m$, and $X \in S$, the following assertions are equivalent*

a) *X is Pareto optimal in S.*
b) *X_j is efficient in S_j for all $j \in M$.*

We may also want to consider weak Pareto optimality (WPO) defined as follows: $X \in S \subseteq \Re^{m \times n}$ is *weakly Pareto optimal* in S, if $\nexists Y \in S$, such that $Y > X$. The analogous equivalence to Proposition 1 holds for weak Pareto Optimality by replacing condition b) by X_j is weakly efficient in S_j for some $j \in M$. Nevertheless, for the class of strictly comprehensive problems we are addressing, as $e(S_j) = we(S_j)$, a global outcome is weakly Pareto optimal if and only if it allocates efficiently the benefits of at least one issue.

2.2 Monotonicity

Monotonicity, a crucial property for an important family of classical solutions, will also turn out very useful in our approach to global solutions:
Monotonicity (MON): A solution φ is *monotonic* if and only if $\varphi(T', d') \leqq \varphi(T, d)$ for all $(T', d'), (T, d) \in \mathcal{B}$, with $d = d'$ and $T' \subseteq T$.

Thomson and Myerson (1980) propose a general family of solutions with monotonicity properties, the monotone path solutions.[4] A *monotone path* (in $\Re^n_+$), G, is defined as the image of a function $\psi : \Re_+ \to \Re^n_+$ where $\psi(0) = 0$, ψ_i is continuous and nondecreasing for all $i \in N$, and $\sum_{i=1}^n \psi_i$ is increasing. The *monotone path solution* relative to the monotone path G, φ^G, is defined as $\varphi^G(T, d) = G \cap e(T)$.

Note that with the present definition monotone path solutions are well defined on convex, compact classical bargaining problems (T, d), provided that $T \subseteq \Re^n_+$ is strictly 0-comprehensive and $d \geqq 0$. Note also that the results the solution yield do not depend on the disagreement point, but when the disagreement point is on the path, the solution is always individually rational.

Monotonicity characterizes the family of monotone path solutions, see Thomson and Myerson (1980). For completeness we sketch a proof of this characterization.

Proposition 2 *A classical solution φ in $\mathcal{B}_0$ is monotonic if and only if φ is a monotone path solution.*

Proof Necessity is immediate. To prove sufficiency, let φ be efficient and monotonic. For each $t > 0$ consider $T_t \subset \Re^n$, $T_t = \{y \in \Re^n, y \geq 0, \sum_{i=1}^n y_i \leq t\}$. Define $\psi(0) = 0$ and $\psi(t) = \varphi(T_t)$. Clearly, the image of ψ is a monotone path and φ the corresponding monotone path solution. □

[4] Thomson and Myerson (1980) use a version of the monotonicity property which is slightly stronger than ours to characterize strictly monotone path solutions.

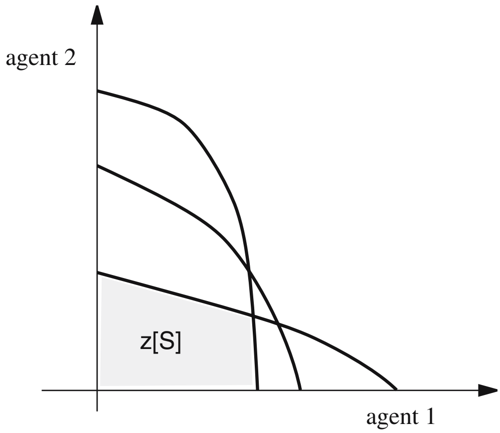

Fig. 1 The set of minimal benefits

3 Maximin bargaining

To begin, we consider situations as that of the Paella example, where the preferences of the bargainers over benefit bundles are determined by the minimum benefit achieved across all the issues.[5]

For these situations the global bargaining problem is very easily addressed as a classical problem via the intersection of the bargaining sets over all the issues. Formally, agent i, has preferences such that $X \succcurlyeq^i_{\min} Y$ if and only if $z^i(X) \geq z^i(Y)$, where $z^i(X) = \min_{1\leq j\leq m}\{x^i_j\}$. Presently, global bargaining problems are reduced to a classical problem whose bargaining set is the intersection of the bargaining sets for all the issues. Note that compactness and comprehensiveness are closed under intersection. Let $z : \Re^{m\times n} \to \Re^n$ be defined as $z(X) = (z^1(X), z^2(X), \dots, z^n(X))$. The function z maps the multiple issue bargaining set S into a set $z[S] \subset \Re^n$, $z[S] = \{z(X) \in \Re^n, X \in S\}$. It is immediate that the comprehensiveness of S_j implies that $z[S] = \cap^m_{j=1} S_j$.

Figure 1 displays $z[S]$ for a two-person, three-issue bargaining problem.

Consider the order relation in $\Re^{m\times n}$, $X, Y \in \Re^{m\times n}$, $X \succcurlyeq_{\min} Y$, if $z(X) \geq z(Y)$. Efficient outcomes under the maximin criterion are straightforward:[6]
Maximin Efficiency (MEF): An outcome $X \in S \subseteq \Re^{m\times n}$ is *maximin efficient* if $\nexists Y \in S$, such that $Y \succcurlyeq_{\min} X$.

It is easy to see that maximin efficiency implies weak Pareto-optimality. In general, Pareto optimality may fail. The following follows immediately:

Lemma 3 *$X \in S$ is MEF if and only if $z(X) \in e(\cap^m_{j=1} S_j)$.*

Our next lemma observes that under strict comprehensiveness of the sets S_j, if $y \in \Re^n$ lies on the efficient frontier of $\cap^m_{j=1} S_j$, there is at least one issue j for which y lies on the efficient frontier of the bargaining set S_j.

[5] This is also the set up of Bossert et al. (1996).

[6] This is a strong version of the *efficiency under uncertainty* in Bossert et al. (1996).

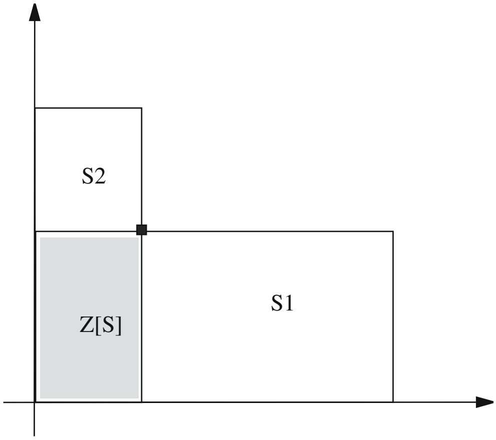

Fig. 2 Efficient solution in $z[S]$

Lemma 4 *If* $y \in \Re^n$, $y \in e(\cap_{j=1}^m S_j)$, *then* $y \in e(S_j)$ *for at least one issue.*

Proof Let $y \in e(\cap_{j=1}^m S_j)$ and suppose that for each j, there exists $\bar{y}_j \in S_j$ such that $\bar{y}_j \geq y$. It follows from the strict comprehensiveness of S_j that for all j, $\exists z_j \in S_j$ with $z_j > y$. Now, for each $i \in N$, denote $\varepsilon^i = \min_j\{z_j^i - y^i\}$ and consider $\bar{y}^i = y^i + \varepsilon^i$. In this situation $\bar{y} \leq z_j$ for all j and therefore, as a consequence of the comprehensiveness of S_j, $\bar{y} \in S_j$ for all j. It follows that $\bar{y} \in \cap_{j=1}^m S_j$ and $\bar{y}^i > y^i$, which is a contradiction to $y \in e(\cap_{j=1}^m S_j)$. □

Note that the full strength of strict comprehensiveness is necessary for this result, as can be seen in Fig. 2, where a global result that is minimax efficient does not allocate efficiently any of the issues.[7]

The following result characterizes maxmin efficient outcomes.

Proposition 5 $X \in S$ *is MEF if and only if* $\exists k \in M$ *such that* $X_k \in e(S_k)$ *and* $X_j \geqq X_k$ *for all* $j \in M$.

Proof If $X \in S$ is MEF, it follows from Lemmas 3 and 4 that $z(X) \in e(S_k)$ for some $k \in M$. Suppose that $X_k \neq z(X)$, then, as $X_k \geqq z(X)$, $X_k \geq z(X)$ holds, what contradicts the efficiency of $z(X)$ in S_k.

Conversely, when $X_j \geqq X_k$ for all $j \in M$, then $z(X) = X_k \in e(\cap_{j=1}^m S_j)$ holds, and therefore $X \in S$ is maximin efficient. □

This result is related to Lemma 1 in Bossert and Peters (2002). The condition $\exists k \in M$ such that $X_j \geqq X_k$ for all $j \in M$ is a generalization to the case of more that two issues of the property of domination,[8] but it also states that in order to be maximin efficient, a global result need not be efficient with respect to all the issues.

[7] Nevertheless, if the assumption is relaxed to comprehensiveness an analogous result can be established obtaining weak Pareto-optimality.

[8] See also Thomson and Myerson (1980).

To generate global solutions that satisfy MEF simply define the global solution $F_\varphi^{\min}$ as $F_\varphi^{\min}(S)_k = \varphi(z[S])$ for all $k \in M$. Denote by F_φ^{sep} the global solution obtained from a classical solution φ in each one of the issues. This is, for $S = S_1 \times \cdots \times S_m$, $F_\varphi^{\text{sep}}(S) = (\varphi(S_1), \ldots, \varphi(S_m))^t$. We want to identify what classical solutions induce the same maximin results under separate or global bargaining, i.e. what φ satisfy the following property:
Separate-global maximin equivalence (SGMEQ): A solution φ satisfies *Separate-Global maximin equivalence* when $\varphi(z[S]) = z(F_\varphi^{\text{sep}}(S))$ for all $S \in \mathcal{GB}_0$.

Next we establish the equivalence between MON and SGMEQ.

Proposition 6 *Let φ be a classical solution in $\mathcal{B}_0$, φ verifies SGMEQ if and only if satisfies MON.*

Proof Consider a classical solution φ that satisfies SGMEQ, and therefore, for $S_1, \ldots, S_m \in \mathcal{B}_0$, $\varphi(z[S]) = z((\varphi(S_1), \ldots, \varphi(S_m))^t)$ holds, and assume φ does not satisfy MON, i.e. there exist $T_1, T_2 \subseteq \Re^n_+$ such that $T_1 \subseteq T_2$, and $i \in N$ such that $\varphi(T_1)^i > \varphi(T_2)^i$. Consider the global bargaining problem $T = T_1 \times T_2 \in \mathcal{GB}_0$. In this case, $z[T] = T_1$ and it follows that $\varphi(z[T]) = \varphi(T_1)$. On the other hand, $z^i((\varphi(T_1), \varphi(T_2))^t) = \min\{\varphi(T_1)^i, \varphi(T_2)^i\} = \varphi(T_2)^i < \varphi(T_1)^i = \varphi(z[T])^i$, contradicting SGMEQ.

Conversely, let φ satisfy MON and $S \in \mathcal{GB}_0$, as $z[S] \subseteq S_j, j \in M$, it follows that for all $i \in N$, $\varphi^i(z[S]) \leq \varphi^i(S_j)$, for all $j \in M$ and, therefore, $\forall i \in N, \varphi^i(z[S]) \leq \min_j\{\varphi^i(S_j)\} = z^i(\varphi(S_1), \ldots, \varphi(S_m))$.

If SGMEQ fails $\exists i \in N$ such that $\varphi^i(z[S]) < \min_j\{\varphi^i(S_j)\}$. From the efficiency of φ follows that $\varphi(z[S])$ is in the efficient frontier of at most a set S_j, for this $j \in M$, $\varphi(S_j)$ is also efficient, and $\varphi^i(z[S]) < \varphi^i(S_j)$, hence $\exists k \in N$ such that $\varphi^k(z[S]) > \varphi^k(S_j)$, contradicting MON. □

Propositions 2 and 6 combined provide the following characterization of the monotone path solutions.

Theorem 7 *A solution φ in $\mathcal{B}_0$ verifies SGMEQ if and only if $\varphi = \varphi^G$, where G is a monotone path.*

4 The leximin approach to global bargaining

For a vector $a \in \Re^m$, let $r(a) \in \Re^m$ be the vector obtained by reordering the components of a in increasing order. For $a, b \in \Re^m$, denote $a >_{\text{lex}} b$ if there is $k = 0, \ldots, m-1$ such that $r_i(a) = r_i(b)$ for $i = 1, \ldots, k$, and $r_{k+1}(a) > r_{k+1}(b)$. Denote $a \geqslant_{\text{lex}} b$ if $a >_{\text{lex}} b$ or $r(a) = r(b)$.

We say that player i has leximin preferences over the global results, and denote $\succcurlyeq^i_{\text{lex}}$, if for $X, Y \in \Re^{m \times n}$

$$X \succcurlyeq^i_{\text{lex}} Y \Leftrightarrow X^i \geqslant_{\text{lex}} Y^i$$

In contrast to the case of maximin preferences, leximin preferences can not be represented by a utility function[9] and therefore it is not possible to reduce the global problem to a classical bargaining problem on the utilities.

We now define a lexicographical ordering relation in $\Re^{m\times n}$ based on the successive minimum values attained by the agents. Given $X \in \Re^{m\times n}$, consider the $m \times n$ matrix $Z(X)$, constructed as follows. For each column X^i—the vector of benefits of player i—reorder its components in increasing magnitude; this reordered vector is the ith column of matrix $Z(X)$. Thus, the first row of $Z(X)$, $z_1(X)$, contains the lowest element of each column of matrix X. The second row, $z_2(X)$, contains the second lowest element of each column of matrix X. In general, the elements of $z_k(X)$ are the k-th lowest element of each column of matrix X. We say that $X \succcurlyeq_{\text{lex}} Y$, if $z_k(X) \geq z_k(Y)$ for the first row, k, such that $z_k(X) \neq z_k(Y)$.

The collective dominance relation $\succcurlyeq_{\text{lex}}$ reduces to the leximin order in $\Re^m$ for $n = 1$, but it does not define a complete order in $\Re^{m\times n}$. The following result states that, as in the case of m-dimensional vectors, the dominance relation $\geq$ among matrices is stronger than $\succcurlyeq_{\text{lex}}$.

Lemma 8 *Let $X, Y \in \Re^{m\times n}$. If $X \geq Y$, then $X \succcurlyeq_{\text{lex}} Y$.*

Proof Since $X \geq Y$, it is clear that $Z(X) \geq Z(Y)$, and therefore, for the first row, k, such that $z_k(X) \neq z_k(Y)$, $z_k(X) \geq z_k(Y)$ holds. □

The relationship between $\succcurlyeq_{\min}$ and $\succcurlyeq_{\text{lex}}$ is immediate.

Lemma 9 *Let $X, Y \in \Re^{m\times n}$. If $X \succcurlyeq_{\min} Y$, then $X \succcurlyeq_{\text{lex}} Y$.*

Denote by $\succ^i_{\text{lex}}$, the asymmetric part of $\succcurlyeq^i_{\text{lex}}$. The following result is also straightforward.[10]

Lemma 10 *Let $X, Y \in \Re^{m\times n}$. If $X \succcurlyeq^i_{\text{lex}} Y$ for all $i \in N$, and $X \succ^i_{\text{lex}} Y$ for some $i \in N$, then $X \succcurlyeq_{\text{lex}} Y$.*

Next we define the concept of efficiency that applies to global bargaining problems where the agents preferences are leximin.
Leximin Efficiency (LEF): A feasible outcome $X \in S \subseteq \Re^{m\times n}$ is *leximin efficient* if there is not another $Y \in S$ such that $Y \succcurlyeq_{\text{lex}} X$.

LEF is clearly stronger than MEF. Whereas MEF implies weak Pareto optimality, LEF implies strong Pareto optimality as a consequence of Lemma 8. A characterization of LEF outcomes is next.

Proposition 11 *$X \in S$ is LEF if and only if for all $k \in M$, $X_k \in e(S_k)$, and for all $j, k \in M$, either $X_j \geqq X_k$ or $X_j \leqq X_k$.*

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[9] See, for instance, Moulin (1988).

[10] It is easy to check that the converse is not true.

Proof Firstly, if for all $k \in M$, $X_k \in e(S_k)$ and for all $j, k \in M$, either $X_j \geqq X_k$ or $X_j \leqq X_k$, let π be the issue index permutation such that $X_{\pi(j)} \leqq X_{\pi(j+1)}, j = 1, \ldots, m-1$. It follows that $z_k(X) = X_{\pi(k)}$ for all $k \in M$. Suppose to the contrary, that X is not LEF, then there exists $Y \in S$, such that for some $k, k = 1, \ldots, m$, $z_j(Y) = z_j(X) = X_{\pi(j)}$ for all $j = 1, \ldots, k-1$, $z_k(Y) \geq z_k(X) = X_{\pi(k)}$. It follows from Lemma 4 and from the efficiency of X_k in S_k that, as $z_1(Y) \in \cap_{j\in M} S_j$ and $z_1(X) = X_{\pi(1)} \in e(\cap_{j\in M} S_j)$, $Y_{\pi(1)} = X_{\pi(1)}$ holds. Analogously, $z_2(Y) = X_{\pi(2)} \in \cap_{j\neq 1} S_{\pi(j)}$ and necessarily, $X_{\pi(2)} = Y_{\pi(2)}$ and for $j = 1, \ldots, k-1$ $X_{\pi(j)} = Y_{\pi(j)}$. For $j = k$, $z_k(Y) \geq z_k(X) = X_{\pi(k)} \in e(\cap_{j\neq 1,\ldots,k-1} S_{\pi(j)})$, $z_k(Y) \in \cap_{j\neq 1,\ldots,k-1} S_{\pi(j)}$, which is in contradiction with the efficiency in the intersection.

Conversely, if X is LEF, it is easy to see that $z_1(X) \in e(\cap_{j\in M} S_j)$ and it follows that there exists $\pi(1)$, such that $z_1(X) = X_{\pi(1)} \in e(S_{\pi(1)})$ and hence $X_j \geqq X_{\pi(1)}$, for all $j \in M$. If $z_2(X) = z_1(X)$, then $X_{\pi(2)} = X\pi(1) \in e(S_{\pi(2)})$, in other case $z_2(X) \in \cap_{j\neq\pi(1)} S_j$, and applying recursively the reasoning the result follows. □

The relationships between the different concepts of efficiency considered so far and their characterizations is summarized as:

Leximin efficient $\forall j \in M, X_j \in e(S_j)$ $\exists \pi, X_{\pi(k)} \leqq X_{\pi(k+1)}, \forall k = 1, \ldots, m-1$	$\Longrightarrow$	Pareto optimal $\forall j \in M, X_j \in e(S_j)$
$\Downarrow$		$\Downarrow$
Maximin efficient $\exists j \in M, X_j \in e(S_j)$ $X_j \leqq X_k, \forall k \in M$	$\Longrightarrow$	Weakly Pareto optimal $\exists j \in M, X_j \in e(S_j)$

5 Global bargaining resolving issues step by step

Next we propose a family of global solutions that select LEF outcomes.

Let $S = S_1 \times \cdots \times S_n$ where each S_j is strictly comprehensive and consider a classical solution, φ. We define the solution, F_φ, for global bargaining problems $(S, 0) \in \mathcal{GB}_0$, as follows: $F_\varphi : \mathcal{GB}_0 \to \Re^{m\times n}$, with $F_\varphi(S) = X^* \in S$ where $X^* \in S$ is obtained by the following procedure:

- *Step 0:* Let $d(0) = 0, I(0) = M, k = 1$.
- *Step k:* For $j \in \{h \in I(k-1), \varphi(\cap_{t\in I(k-1)} S_t, d(k-1)) \in e(S_h)\}$,
  $X_j^* = \varphi(\cap_{t\in I(k-1)} S_t, d(k-1))$.
  $I(k) = I(k-1) \setminus \{h \in M, \varphi(\cap_{t\in I(k-1)} S_t, d(k-1)) \in e(S_h)\}$.
  - If $I(k) = \{\emptyset\}$, then $F_\varphi(S) = X^*$.
  - If $I(k) \neq \{\emptyset\}$, then $d^i(k) = \varphi_i(\cap_{t\in I(k-1)} S_t, d(k-1))$ for all $i \in N$.

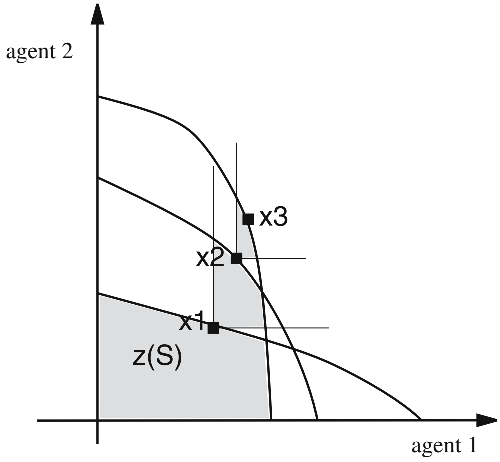


Fig. 3 Step by step bargaining procedure

The procedure uses a classical solution to sequentially select the agents' benefits for the issues. In each step, the classical solution is applied to the set of minimum benefits of the issues that are still unresolved, fixing the level of benefits for at least one issue. Then a new classical bargaining problem is considered, whose disagreement point consists of the levels of benefit assigned to the agents in the issue(s) fixed by the previous step.

It follows from Lemma 4 that at every step at least one issue is resolved and therefore, the values of X^* are selected at most in m steps, that is, $I(k) = \{\emptyset\}$ for some $k \leq m$. Note that if φ is defined uniquely, then F_φ is also defined uniquely. The procedure is illustrated in Fig. 3.

Proposition 12 *A global bargaining solution F_φ selects a LEF outcome.*

Proof The result follows from the construction of F_φ and the characterization of LEF outcomes established in Proposition 11. □

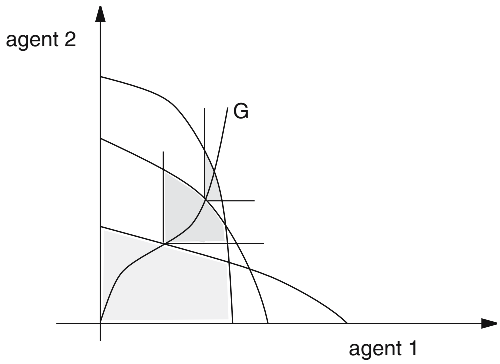
In general, the solution to the global problem induced by a solution φ, F_φ does not coincide with the vector of solutions obtained if bargaining takes place separately in each issue. If it does, we will say that φ verifies step by step equivalence.

Step by step equivalence (SSEQ): A classical solution φ satisfies *step by step equivalence* if and only if $F_\varphi(S) = F_\varphi^{sep}(S)$.

This axiom has the flavour of the step by step axiom in Kalai (1977). However, while Kalai addresses additive utility bargaining problems where the bargaining set expands as new issues are added to the problem, SSEQ concerns global bargaining problems where the outcomes consist of the results obtained for each issue, which are valued in a leximin preference domain.

To conclude we establish that SSEQ also characterizes the family of monotone path solutions. Figure 4 is an illustration.

Fig. 4 Monotone path solution



Theorem 13 *A classical solution φ satisfies SSEQ, i.e. $F_\varphi(S) = F_\varphi^{sep}(S)$, if and only if $\varphi = \varphi^G$, where G is a monotone path.*

Proof To proof that if $\varphi = \varphi^G$ with G a monotone path, then $F_\varphi(S)_k = \varphi(S_k)$, first note that $\varphi^G(T,0) = \varphi^G(T,d)$ for all $d \in G \cap T$. As F_φ is leximin efficient, there exists π such that $F_\varphi(S)_{\pi(k)} \leqq F_\varphi(S)_{\pi(k+1)}$, for all $k = 1,\ldots,m-1$. In this situation, $F_\varphi(S)_{\pi(1)} = \varphi(\cap_{j\in M} S_j) \in e(S_{\pi(1)})$. Besides, $\varphi(S_{\pi(1)}) \in e(S_{\pi(1)})$, and as a consequence of the monotonicity of φ, $\varphi(\cap_{j\in M} S_j) \leqq \varphi(S_{\pi(1)})$, therefore necessarily $\varphi(\cap_{j\in M} S_j) = \varphi(S_{\pi(1)})$. In the second step of the construction of F_φ, the disagreement point is $\varphi(S_{\pi(1)})$ which is on the path. Now, $F_\varphi(S)_{\pi(2)} = \varphi(\cap_{j\neq\pi(1)} S_j, \varphi(S_{\pi(1)})) = \varphi(\cap_{j\neq\pi(1)} S_j, 0) \in e(S_{\pi(2)})$, and also $\varphi(S_{\pi(2)}) \in e(S_{\pi(2)})$. It follows that $\varphi(S_{\pi(2)}) = F_\varphi(S)_{\pi(2)}$. Analogously the same result is obtained for each issue.

Conversely, from the leximin efficiency of the solution F_φ it follows that there exists π such that $\varphi(S_{\pi(k)}) \leqq \varphi(S_{\pi(k+1)})$ for $k = 1,\ldots,m-1$. For the first issue solved, $F_\varphi(S)_{\pi(1)} = \varphi(S_{\pi(1)}) \leqq \varphi(S_{\pi(k)})$ holds for all k, and therefore, $z(\varphi(S_1),\ldots,\varphi(S_m)) = F_\varphi(S)_{\pi(1)}$.

In addition $\varphi(z[S]) = F_\varphi(S)_{\pi(1)}$, and from Theorem 7 it follows that φ is a monotone path solution. □

As a consequence of this last result, if classical solutions are restricted to those that satisfy homogeneity, i.e. $\varphi(\gamma(S)) = \gamma\varphi(S)\ \forall\gamma > 0$, then solutions verifying SSEQ are the proportional solutions. If symmetry is also required the unique solution satisfying this separability axiom is the egalitarian.

It is worth pointing out that SSEQ is a separability condition analogous to the separate/global equivalence condition in Ponsatí and Watson (1997), where global bargaining problems with additive utilities are addressed. In our set up, symmetry, homogeneity and SSEQ characterize the egalitarian solution, whereas, for additive utilities, symmetry, homogeneity and Ponsatí and Watson's separate/global equivalence characterizes the symmetric utilitarian solution. Binmore (1984) characterizes the Nash solution for the case of multiplicative utilities with a related separability condition.

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