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Chores[1]

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We analyze situations where the provision of each of c public goods must be voluntarily assumed by exactly one of n private agents in the absence of transfer schemes or binding contracts. We model this problem as a complete information, potentially infinite horizon game where n agents simultaneously wage c wars of attrition. Providing a public good commits an agent not to take on the provision of another public good for a fixed period. We explore the strategic trade-offs that this commitment ability and the multiplicity of tasks provide. Subgame perfect equilibria (SPEs) are characterized completely for games with two agents and two public goods. For games with two identical agents and $c > 1$ identical public goods, we establish that an equilibrium that yields a surplus-maximizing outcome always exists and we provide sufficient conditions under which it is the unique equilibrium outcome. We show that under mild conditions, the surplus-maximizing SPE is the unique symmetric SPE. *Journal of Economic Literature* Classification Number: H41, C72, D13. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

The welfare of nations, families, and institutions relies on the voluntary provision of public goods by individual members of the group. This private provision of public goods often takes place in the absence of enforceable contracts or of transfer schemes that would compensate individuals for the private costs that they bear in providing collective benefits. The allocation of domestic chores within a household is an obvious example of such a situation: a number of chores must be allocated among household members, and for each chore, the individual undertaking it is the one who alone bears the cost of performing the service, while everyone benefits from the chore being done. Professional partnerships and cooperative firms face task-sharing problems of a similar nature. Research teams must allocate tasks to complete a project, clinics must allocate patients among physicians, clients must be assigned to lawyers within a firm, and so on. In a world of incomplete contracts, even firms that offer their employees detailed labor contracts can face task-sharing problems in the workplace. Countries jointly participating in military exercises or peace-keeping efforts must coordinate their operations. In federal states with different levels of government (e.g., federal, provincial, municipal), task-sharing problems can arise when different levels of government share responsibilities in a given sphere of social or economic policy.

In this paper we study the behavior of a community of strategic agents who must privately provide many indivisible public goods in the absence of transfer schemes or contracts.[2] We assume that agents prefer to have public goods that are freely provided by someone else rather than to provide them themselves. However, each individual is willing to bear the cost of providing the public good herself if the alternative is to do completely without it. Under these assumptions the situation in which agents wait for someone to volunteer for the provision of a *single* indivisible public good is an all or nothing conflict. It has been modeled using a war of attrition, where each individual decides how long she will wait for someone else to volunteer to provide the public good before volunteering to provide it herself.

Bliss and Nalebuff (1984) and Bilodeau and Slivinski (1996) present suggestive analyses in this one-public-good context. Bliss and Nalebuff (1984) emphasize the role played by private information about each agent's cost of providing the good. Time in their model is a screening device. The wait

[2]Situations in which agents decide the amount of their contribution in a continuous fashion are considered elsewhere in the private provision of public goods literature. In particular, Bergstrom *et al.* (1986) analyze noncooperative contribution games in which strategies are donations limited by the wealth of each player. They address the effects of wealth redistribution on aggregate donations.

that an agent will endure in the war of attrition is an increasing function of her private cost of providing the public good and thus, in equilibrium, the person with the lowest cost of provision volunteers first. The war of attrition game analyzed in Bilodeau and Slivinski (1996) features a unique equilibrium in which the agent with the highest benefit/cost ratio volunteers immediately. Thus, in both treatments of the single good case, the agent who should volunteer from a social standpoint actually does so in equilibrium; further, the equilibrium outcome either features no delay, or a delay just sufficient for optimal assignment of the public good provision to take place. Decentralized decision-making, in which agents simply wait for someone to volunteer, achieves an efficient provision of the public good.

In most contexts in which one public good must be voluntarily provided by one private agent within a group, it is natural to think that other public goods must be provided in the same manner. In a university department—an instance of a professional partnership—being chair is but one example of a service for which the costs are borne privately by the individual who volunteers, while most benefits of the service are enjoyed by everyone. Typically, for the same group of agents, the task of being graduate program director and the task of being dean have the same characteristics. Similarly, a group of individuals living together face the assignment of not just one but many household chores. Furthermore, undertaking one task most often excludes doing another at the same time. An individual volunteering to be graduate director ensures that for the length of her term, she will not be department chair. Similarly, when different levels of government share responsibility for the provision of certain social services, if a given level of government provides one service, a resource constraint may exclude the possibility of providing another service during the same budget period.

These observations inspire our approach to the modeling of agents volunteering for the provision of many public goods. Agents in our model can only do one thing at a time, so that an individual who volunteers for one chore thereby makes a credible commitment not to take another chore for a set period. We analyze a complete information, potentially infinite horizon game in which n agents share $c > 1$ chores (i.e., privately provide c public goods). In each period, agents are asked, in random order, either to select a task or to wait and do nothing. A player who selects a task is not allowed to leave it unfinished, and the costs and benefits of the task are realized once it is completed.

We explore the strategic trade-offs that the ability for commitment and the multiplicity of tasks provide, and the efficiency and fairness properties of subgame perfect equilibrium (SPE) outcomes. We first provide a complete characterization of the set of SPEs for general *two-person, two-task* (2×2) games. We find that whether a game features specialization (one agent does both chores) or task-sharing, the second task is always taken as soon as possible. Delay in taking a task only occurs at the beginning of the game. Furthermore, behavior at the first concession can be characterized using general results on the war of attrition.

When transfers are not possible as in the present context, appealing normative prescriptions should promote fairness as well as efficiency. Fairness is confronted to efficiency in the case of a single public good: efficiency requires that one agent volunteer immediately, but then one agent unfairly bears the entire cost of providing the good. As the number of public goods increases, the opportunities to combine fairness with efficiency are enhanced relative to the all or nothing situation of the war of attrition. The promise that, after the first chore is assigned, others will follow by performing the remaining chores is a reward that can convince an agent to be the first to volunteer. In 2×2 games, allocations in which each player does one task and in which there is no delay are especially appealing. Our findings are that these allocations arise as SPEs under very general conditions. Further, we provide conditions under which the unique SPE yields such an allocation.

We extend our results to games with two identical players and (many) *equivalent* tasks (APET games). For APET games an allocation is surplus-maximizing if and only if all tasks are taken as quickly as possible. This requires that agents alternate in volunteering and never delay taking any task; as a result, tasks are distributed among agents as evenly as is feasible. Thus, for APET games, promoting fairness (or equal burdens) is tantamount to attaining a surplus-maximizing allocation. We show that surplus-maximizing SPEs always exist and we give sufficient conditions under which the surplus-maximizing SPE is the unique SPE. Moreover, under mild conditions that include simply having a large number of tasks, there is a unique SPE in *symmetric strategies*, and this equilibrium is the surplus-maximizing SPE. We briefly indicate how our results generalize for games with $n > 2$ agents.

The paper is organized as follows. Section 2 presents the model. In Section 3, we briefly review the war of attrition and the one-chore game. In Section 4, we characterize SPEs for general 2×2 games. Section 5 focuses on APET games with $c \geq 2$. Section 6 presents some extensions. Section 7 concludes.

2. MODEL AND DEFINITIONS

A set of individuals $N = \{1, 2, \ldots, n\}$ must share a set of chores $C = \{1, 2, \ldots, c\}$. The allocation of chores takes place over time $t = 0, 1, 2, \ldots$. An individual j taking chore k in period t is busy doing this chore for $\tau(j, k) - 1$ additional periods; therefore, benefits from task k accrue to all agents at time $t + \tau(j, k)$, with $\tau(j, k) \geq 3$. Agent j discounts at a rate δ_j, $0 < \delta_j < 1$, per period. For any given chore, agents prefer someone else to do the chore rather than doing it themselves, but they prefer to take the chore rather than leaving the job forever undone. Letting $\delta_j^t\ Y(j, i, k)$ be the (net) discounted benefits that accrue (at $\tau(i, k) + t$) to agent j when some other agent $i \neq j$ performs task k starting in period t, and letting δ_j^t $X(j, k)$ be the (net) discounted benefits that accrue to j when she performs the chore herself, we assume that

$$Y(j, i, k) > X(j, k) > 0 \qquad \forall\, j, k, i \neq j.$$

We restrict our attention almost exclusively to two player games. In this context, we can simply write $Y(j, k)$ since if j is not doing task k, then it must be i who is doing the task.

Chores are allocated as individuals play the following noncooperative game. In each period, a permutation of unoccupied players is selected at random. The first player in the permutation chooses either to pass and remain unoccupied, or to concede by taking one of the available chores. If she takes a chore, the game moves on to the next period.[3] Since she is busy for at least three periods, she commits not to take another chore for at least the next two periods.[4] If instead she chooses to remain unoccupied, then the next player in the permutation chooses either to take one of the available chores or to remain unoccupied, and so on. If all unoccupied players pass in period t, the game moves on to the next period. This continues forever, or until all chores have been performed. In what follows, we use #1 (#2, etc.) to refer to the player picked first (second, etc.) in the permutation.[5]

The game just defined allows for diversity of both individuals' characteristics and tasks. The benefits that accrue to agents when tasks are performed, as well as the time commitment involved in a task, can vary with the task and with the identity of the individual. Time preferences are also player-specific. Defining simpler games, games that restrict the extent to which player and/or tasks are diverse, greatly simplifies the analysis.

[3]Using an alternative formulation that allows for more than one task to be started in a given period (i.e., after player i chooses a task in period t, player j can start another in period t) does not change the substance of our results. Our formulation has the advantage of mimicking a continuous time setup since it precludes any two players from taking a task at a given point in time. We therefore expect our results to extend without much difficulty to a continuous time formulation.

[4]A task must last at least two periods to constitute a commitment not to undertake another task. Our results hold for the case where $\tau(j, k) = 2$, although in some cases the results become trivial.

[5]In our context for which time commitments to tasks would plausibly be observable, a formulation with sequential moves seems more appropriate. We remark on the consequences of this sequential formulation in what follows.

We define a game to have *anonymous players* (AP) as one in which permuting the labels of the players does not change the strategic nature of the game: $Y(j,i,k) = Y(k)$, $X(j,k) = X(k)$, $\delta_j = \delta$, $\tau(j,k) = \tau(k)$, $\forall j,k$. All elements of the game potentially depend on the particular task under consideration but not on the identities of the players. This can be interpreted to mean the following: (1) Agents have the same preferences over tasks; i.e., once i takes task k at time t, benefits do not depend on the identity of the recipient ($Y(j,i,k) = Y(i,k), X(j,k) = X(k)$). (2) Agents have the same time preference ($\delta_j = \delta$). (3) Agents have the same ability to generate social surplus; i.e., given that $i \neq j$ performs task k, benefits are invariant to the identity of the agent who performs the task ($Y(i,k) = Y(k)$). (4) All agents get the same commitment value from a given task ($\tau(j,k) = \tau(k)$). We also define a game to feature *equivalent tasks* (ET) as one in which permuting the labels for the tasks does not change the strategic nature of the game, $Y(j,i,k) = Y(j,i)$, $X(j,k) = X(j)$, $\tau(j,k) = \tau(j)$, $\forall j,k$. Tasks are interchangeable; the benefits and the time commitment associated with each task only depend on the identity of the players. An *APET game* combines both anonymous players and equivalent tasks, $Y(j,k) = Y$, $X(j,k) = X$, $\delta_j = \delta$, $\tau(j,k) = \tau$, $\forall j,k$. In addition to analyzing general games in the two-player two-task context, we also analyze APET games in which there are many chores.

A *history of the game at time t* is the ordered list of all the actions that each player has chosen from date 0 to $t-1$, remaining idle or undertaking a task k, where each task k appears as an action taken at no more than one date. For each history of the game, the payoff to player i at time t is the sum of the payoffs obtained from all the tasks that are completed at t or earlier. For each history at t and each sequence of players preceding player i at time t, that is, for each subgame, a *strategy for player i* selects a feasible action. A *subgame perfect equilibrium* is a strategy profile such that for each player, the action that she selects at each subgame maximizes her payoff given the strategies of the other players. For APET games, we sometimes restrict attention to symmetric strategies. In a symmetric strategy profile, permuting the labels of the players leaves the strategy profile unchanged. A *symmetric subgame perfect equilibrium* (SSPE) is a strategy profile that is a SPE and that is symmetric.

An *allocation of tasks* is given by the function Γ: $C \longrightarrow [N \cup \varnothing \times \{0,1,2,\ldots\} \cup \varnothing]^c$ where $\Gamma(1,\ldots,c) = (\gamma(1),\ldots,\gamma(c))$. If task h is undertaken by j at time t, then $\gamma(h) = (j,t)$ while if h is left undone, $\gamma(h) = (\varnothing,\varnothing)$. The allocation is *feasible* iff it follows the rules of the game. Only one task can be assigned at any given time t ($\gamma(k) = (j,t) \Rightarrow \gamma(h) \neq (i,t), h \neq k$). Further, given that a player j undertakes a task k at time t, another task h cannot be assigned to this same player before she has completed task k ($\gamma(k) = (j,t) \Rightarrow \gamma(h) \neq (j,t')$

for $k \neq h$, and $\forall t'$ such that $t \leq t' \leq t + \tau(j,k) - 1$). We say that a task allocation features *task-sharing* when tasks are divided among the players as equally as possible. That is, with c tasks and n players, each player is assigned c/n tasks if c/n is an integer, or one of the two integers closest to c/n otherwise. When instead one player takes more than her "equal" share of c chores while some other player takes fewer, we will refer to the outcome as featuring some degree of *specialization*.

A feasible allocation Γ' *Pareto dominates* another feasible allocation Γ if and only if for each player, the net benefits are at least as large under Γ' as they are under Γ, and for at least one player, the net benefits from Γ' are strictly greater. A feasible allocation Γ is *Pareto optimal* if no other feasible allocation Pareto dominates it.[6]

Pareto optimality rules out an allocation in which no task is undertaken in period t even though some players are free; indeed, all players would prefer for the player who eventually takes the next task to have done so at an earlier opportunity. It is easy to see that a wide range of task assignments are compatible with Pareto optimality, including in certain circumstances allocations that have one player doing all tasks. Especially for APET games, a notion stronger than Pareto optimality is better at capturing reasonable normative requirements of efficiency and fairness. We thus also define an allocation to be *surplus-maximizing* if it maximizes the sum of benefits for all players. A surplus-maximizing allocation is always Pareto optimal but in our context in which there are no transfers, the converse is not true. Surplus maximization requires that all tasks be completed eventually, and that whenever there are players free, one must volunteer for a task. Moreover, players should perform the tasks for which they have a comparative advantage, and tasks that yield higher social surplus should be performed earlier. In an APET game, varying the assignment of tasks to players has no effect on social surplus; only the assignment of tasks across time matters. In these simpler games, surplus maximization is equivalent to minimizing delay and thus it rules out complete specialization by one player. When n players share $c \geq n$ tasks and tasks are long ($\tau > n$), surplus maximization implies task-sharing.

3. WARS OF ATTRITION AND ONE-CHORE GAMES

When there is only one chore and two players, each player can take only one of two moves at each t, either concede or pass. The game reduces to a

[6]We use an *ex post* notion of allocation and Pareto optimality which is much simpler to define. Our discussion of Pareto optimal allocations would also apply to *interim* and ex ante notions.

simple concession game, the war of attrition. This game is well known and has been extensively studied in the literature.[7] Concession by any player at t terminates the game. Let the payoffs be U_i to the player who concedes and W_j to her opponent, $j \neq i$.

With some abuse of terminology, we use the term *war of attrition* to refer to concession games with one task, without any restriction on the range of payoffs U_i and W_i. Standard treatments of the war of attrition assume that $U_i \leq W_i$ for all i because concession games for which $U_i > W_i$ for some i are strategically trivial: player i in equilibrium concedes as soon as possible. Our interest in the war of attrition is not in studying the game in isolation, but in using it as a building block in our characterization of SPEs for games with two or more tasks. As will become apparent later on, in our context of a multichore game, the full range of payoffs of the war of attrition is relevant. The following proposition thus provides a complete characterization of SPEs for our sequential version of a two-player war of attrition.

PROPOSITION 1. *Consider a two-player war of attrition with payoffs* $((U_1, W_2), (W_1, U_2))$. *The strategy profiles below constitute the complete set of SPEs for the game. Unless otherwise specified, the actions of these strategy profiles are given for each period regardless of the order of the players in the permutation.*

(a) *Given* $U_i > W_i$ *for* $i = 1$ *and* 2, *both players concede*. (b) *Given* $U_i > W_i$ *and* $W_j > U_j > \delta W_j$ *for* $i, j = 1$ *or* 2, $i \neq j$, *i concedes, j passes as* #1 *and concedes as* #2. (c) *Given* $U_i \leq \delta_i W_i$ *for* $i = 1$ *or* 2, *i passes, j concedes* (*the j*-**pure SPE**). (d) *Given* $W_i > U_i > (\delta_i/(2 - \delta_i))W_i$ *for* $i = 1$ *and* 2, *both pass as* #1 *and concede as* #2 (*the* **pure SPE on role**). (e) *Given* $U_i \leq (\delta_i/(2 - \delta_i))W_i$ *for* $i = 1$ *and* 2, *both pass as* #1 *and concede with a given player-specific probability* $p_i \in (0, 1)$ *as* #2 (*the* **mixed SPE on role**). (f) *Given* $\delta_i W_i \geq U_i \geq (\delta_i/(2 - \delta_i))W_i$ *and* $W_j > U_j$ *for* $i, j = 1, 2$, $i \neq j$, *j concedes with a given player-specific probability* $\hat{p}_i \in (0, 1)$ *as* #1 *and concedes with certainty as* #2, *while i passes as* #1 *and concedes with a given player-specific probability* $\hat{p}_j \in (0, 1)$ *as* #2 (*the j*-**mixed SPE**).

The complete proof of Proposition 1 is standard and is omitted for the sake of brevity.[8] We briefly comment on the SPEs. When $U_i > W_i$ for at least one player, player i prefers to concede and thus concession is immediate. With $U_i \leq W_i$ for both players, concession is immediate when players

[7] We refer the reader to Hendricks *et al.* (1988) and Ponsatí and Sákovics (1995) for comprehensive treatments of the two-player war of attrition under complete and incomplete information, respectively; also see Nalebuff and Riley (1985) for an early treatment in an evolutionary context.

[8] The results in the rest of the section follow from this proposition. The proofs for these results are also straightforward.

use pure strategies in equilibrium. In a *pure strategy SPE on identity*, a particular player (say i) is targeted to concede, and the other player passes in the expectation that i will concede. In the *pure strategy SPE on role*, the task assignment is effected through the permutation of players taken at the start of the period. Whoever is chosen to act as #2 concedes. The remaining SPEs with $U_i \leq W_i$ are mixed strategy equilibria in which there is a positive expected delay before the first concession. In the *mixed strategy equilibrium on role*, each player passes as #1 and randomizes between conceding immediately and going on to the next period #2. Each player knows that by waiting until the next period, there is some probability that the other player will be chosen #2 and will choose to concede. In the *mixed strategy SPE on identity*, one player (say i) is targeted to randomize as #1 and to concede for sure as #2, while the other player passes as #1 and randomizes as #2.

Depending on the parameter configuration, there might be one, three, or five SPEs.[9] Figure 1 graphs the possible scenarios given in Proposition 1 while the following corollary pinpoints the circumstances in which the SPE is unique.

COROLLARY 1. *The SPE in the war of attrition is unique and the first concession occurs immediately in equilibrium in each of the following three circumstances*: (i) $U_i \geq \delta_i W_i$ *for* $i = 1$ *and* 2; (ii) $U_i > W_i$ *and* $U_j < \delta_j W_j$ *for* $i, j = 1, 2$ *and* $i \neq j$; (iii) $W_i > U_i > \delta_i W_i$ *and* $U_j < (\delta_j/(2-\delta_j))W_j$ *for* $i, j = 1, 2$ *and* $i \neq j$.

The preceding two results are stated for a general concession game. Our task-allocation game with $n = 2$ and $c = 1$ is a special case of this game in which concession for the unique task k at time t yields $\delta_i^t X(i,k) = \delta_i^t X_i$ to the player who concedes and $\delta_j^t Y(j,k) = \delta_j^t Y_j$ to her opponent. Further, since the task-allocation game assumes that $Y_j > X_j$, the SPEs can be more narrowly characterized as follows.

PROPOSITION 2. *Let* $n = 2$ *and* $c = 1$. *The following are the SPEs of the game*. (a) *The j-pure SPE when* $X_i \leq \delta_i Y_i$, $i = 1$ *or* 2, $i \neq j$. (b) *The mixed on roles when* $X_i \leq (\delta_i/(2-\delta_i))Y_i$, $i = 1$ *and* 2. (c) *The pure on roles when* $X_i > (\delta_i/(2-\delta_i))Y_i$, $i = 1$ *and* 2. (d) *The j-mixed when* $\delta_i Y_i \geq X_i \geq (\delta_i/(2-\delta_i))Y_i$.

COROLLARY 2. *The SPE in the* $n = 2$, $c = 1$ *game is unique when either* (i) $X_i \geq \delta_i Y_i$ *for* $i = 1$ *and* 2, *in which case the pure SPE on roles prevails*; *or*

[9]The set of SPE for the simultaneous move war of attrition has a simpler structure (either i-pure, j-pure, and mixed; or only one of the i, j-pure). We choose an alternating move specification since it facilitates the analysis for $c \geq 2$. Moreover, the richer structure of the set of SPE for $c = 1$ is of no consequence for SPEs when $c \geq 2$.

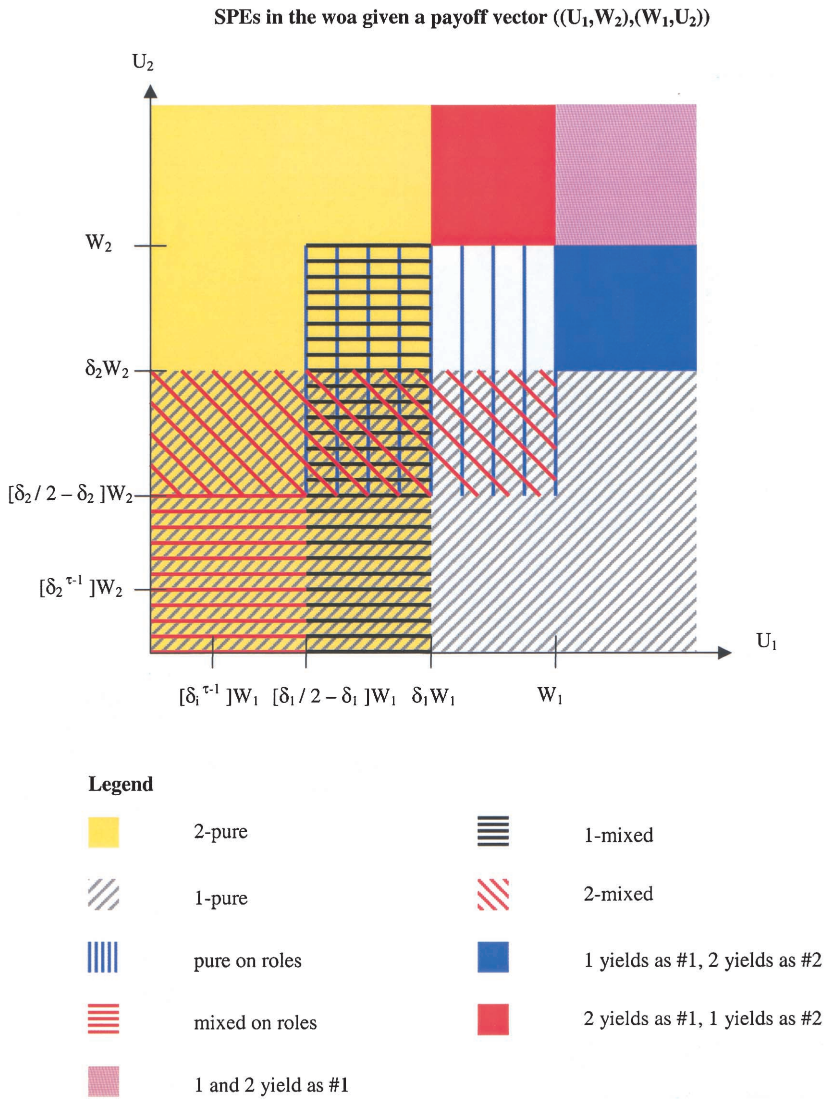

FIG. 1. SPEs in the woa given a payoff vector $((U_1, W_2), (W_1, U_2))$.

(ii) *If* $X_j \geq \delta_j Y_j$, *and* $X_i < (\delta_i/(2-\delta_i))Y_i$, $i, j = 1, 2$ *and* $i \neq j$, *in which case the j-pure SPE prevails.*

For the anonymous player version of this game ($Y_j = Y_i = Y$ and $X_j = X_i = X\ \forall\, j, i \neq j$), the results can be further simplified as follows.

COROLLARY 3. *Let* $n = 2$, $c = 1$ *and assume anonymous players*. (i) *If* $\delta Y \leq X \leq Y$, *then the* **pure SPE on role** *is the unique SPE*. (ii) *If* $\delta Y > X \geq \frac{\delta}{2-\delta} Y$, *then there are five SPEs*: *the pure SPE on role*, *the i-pure SPE*, *the j-pure SPE*, *the i-mixed SPE*, *and the j-mixed SPE*. (iii) *If* $X < \frac{\delta}{2-\delta} Y$, *then there are three SPEs*: *the two pure SPEs on identity*, *and the mixed SPE on roles*.

Surplus-maximization requires that a player takes the chore right away. If the value to an agent of performing the task herself is high enough ($X > \delta Y$) the task is done in period 0. However, in the opposite case for which a player values her opponent doing the chore highly enough, or in the more usual simultaneous-move war of attrition studied elsewhere in the literature, surplus-maximization is attained only if the pure strategy equilibrium on identity is played. In any mixed strategy equilibrium, there is an expected delay before the chore is performed as both players remain idle with positive probability in each period. Hence surplus-maximization is attained only when equilibrium behavior is asymmetric (one player never concedes while her opponent always yields). In contrast, we will see in what follows that in multitask APET games, surplus-maximization and fairness are compatible and closely linked to symmetric strategy profiles.

4. TWO PLAYERS, TWO CHORES

Task-allocation games with two players and $c \geq 2$ chores are still concession games. The game ends when there have been c concessions, that is, when all c tasks have been undertaken. However, in contrast to the one-task context, games with multiple tasks are not always all-or-nothing conflicts. A player can concede on a task while her opponent is busy with another task. To highlight the qualitative features of the strategic interaction in multiple task environments, in this section we analyze carefully the case of two players and two chores.

In the most general sense, two kinds of outcomes are possible in an $n = 2$, $c = 2$ game: a player could do both tasks or each player could do exactly one task. Given a strategy profile, whether the outcome features specialization (one player does both tasks) or task-sharing can depend on the identity of the player who concedes first. For instance, if player i puts relatively more value on the task being done by her opponent, player i may credibly be able to wait for j to do the second task when j concedes on the first task, but player j may prefer to undertake the second task if player i concedes on the first task. Correspondingly, when characterizing SPEs of the game, we will distinguish a *specialization equilibrium* from a *j-specialized equilibrium*. A SPE is a *specialization equilibrium* when the outcome features a single player performing both chores regardless of the identity of the player who concedes first. A SPE is a *j-specialized equilibrium* when the outcome features player j performing both tasks when j concedes first but it features players sharing tasks when i concedes first. When the outcome of an SPE features players sharing tasks regardless of which player concedes first, we will say that the equilibrium is a *task-sharing equilibrium*. For each of these broad classes of outcomes, players could plan to take tasks at the first opportunity or they could plan to delay. If there is no delay and (say) player j is the first to concede by taking task k, then with task-sharing the game ends in period $\tau(j, k)$ or in period $\tau(i, h) + 1$ (whichever is later), while with specialization the game ends in period $\tau(j, k) + \tau(j, h)$. In principle, delay can occur before the first or before the second concession.

In the remainder of the section, we characterize the set of SPEs for the $n = 2$, $c = 2$ game, paying particular attention to circumstances in which tasks are shared without delay. We proceed in three steps. First, we characterize SPE profiles at subgames after the first task has been taken. A direct implication of this characterization is that in equilibrium, delay can only occur before the first task is taken. Second, we provide sufficient conditions that ensure the existence of a unique SPE outcome in which tasks are shared without delay. Finally, we provide a complete characterization of SPEs for the game. Equilibria of the war of attrition, as characterized by Proposition 1, play a fundamental role at each step.

4.1. *The Game after One Task Is Taken*

We consider subgames after the first concession and we characterize the behavior that results from equilibrium play. We characterize in two ways behavior in subgames after one task is taken: by identifying circumstances in which a first concession by a player means that she will do both tasks, and by determining whether there are circumstances in which delay can occur.

Without loss of generality, we assume that player j has taken a first task k in period t. The subgames of interest thus begin in period $t + 1$. From $t + 1$ until $t + \tau(j, k) - 1$, player j is busy and player i is the only player who actively chooses whether to concede on the last task or to pass. Each subgame that starts in period $t + \tau(j, k)$ or later is a war of attrition (woa): both players are unoccupied and each player can concede to end the game. The behavior in subgames starting between $t + 1$ and $t + \tau(j, k) - 1$ depends on expectations regarding play when both players are free in $t + \tau(j, k)$. For a given strategy profile of the two-chore game, we will say that *some woa continuation prevails* for a subgame in which one chore is left and both players are idle when the strategy profile prescribes the *i-pure*, *j-pure*, *i-mixed*, *j-mixed, pure-on-roles, or mixed-on-roles* (see Proposition 1). For example, if after player j takes task k the j-pure woa continuation prevails, then this means that once player j completes task k and once only task h is left, the strategies prescribe that in each subsequent period, player j concedes on task h and player i passes.

For specialization to occur once j has undertaken the first task, it is certainly sufficient for the j-pure woa continuation to prevail and for player i to be willing to wait $\tau(j, k) - 1$ periods for player j to start the second task. If the j-pure woa continuation prevails, player j will always concede at her first opportunity once both players are free and one chore is left. Further, if the game is $SP(j, k)$ which means that

$$X(i, h) < \delta_i^{\tau(j,k)-1} Y(i, h) \qquad \text{for } i \neq j \quad \text{and} \quad h \neq k \tag{1}$$

is satisfied, then player i prefers to wait for player j to finish the first task and start the second task rather than to undertake the second task herself. For obvious reasons, we refer to a game in which $SP(j, k)$ is satisfied for some (j, k) as a game that is *specialization feasible*.

The following lemma characterizes the best reply of player i in subgames when she alone is free. The lemma establishes that the condition $SP(j, k)$ and a j-pure woa continuation are not only sufficient for specialization to occur after j has taken the first task k; they are also necessary.

LEMMA 1. *Suppose that player j takes a first task k at time t. Consider one-chore subgames starting from time t to time $t + \tau(j, k) - 1$. In any strategy profile that constitutes a SPE*:

(I) *If the pure-on-roles, the i-pure, or any mixed equilibrium woa continuation prevails, then at any date $t + m$ $(1 \leq m \leq \tau(j, k) - 1)$ it is a best reply for player i to concede.*

(II) *If the j-pure woa continuation prevails and the game is $SP(j, k)$, then at any date $t + m$ $(1 \leq m \leq \tau(j, k) - 1)$ it is a best reply for player i to pass.*

(III) *If the j-pure woa continuation prevails and the game is not $SP(j, k)$, then at dates $1 \leq m \leq \tau(j, k) - (q - 1)$ player i concedes while at dates $\tau(j, k) - q \leq m \leq \tau(j, k) - 1$ player i passes, where $q \geq 2$ and $\delta_i^{q-1} Y(i, h) > X(i, h) \geq \delta_i^q Y(i, h)$.*

Proof. Let $X_i = X(i, h)$ and $Y_i = Y(i, h)$. If the i-pure woa continuation prevails, then in any subgame in which i is the only active player, i is strictly better off taking the second task than delaying: $X_i > \delta_i X_i$.

If the j-pure woa continuation prevails, player i prefers taking the second task at $t + m$ rather than passing if $X_i \delta_i^m \geq Y_i \delta_i^{\tau(j,k)}$ which is equivalent to

$$X_i \geq \delta_i^{\tau(j,k)-m} Y_i. \tag{2}$$

If (2) does not hold at $m = 1$, then for each date $t + m$ it is better for player i to pass:

$$X_i < \delta_i^{\tau(j,k)-1} Y_i \le \delta_i^{\tau(j,k)-m} Y_i, \qquad 1 \le m \le \tau(j,k) - 1.$$

If (2) holds at $m = 1$, then player i takes the second task at time $t + 1$ because she is unwilling to wait $\tau(j,k) - 1$ periods for player j to become free and take the remaining task. If (2) holds at $m = 1$ and $X_i \ge \delta_i Y_i$, then player i takes the remaining chore at each date $t + m$. If (2) holds at $m = 1$ and $X_i < \delta_i Y_i$, then there is some integer q such that $q \ge 2$ and $\delta_i^{q-1} Y_i > X_i \ge \delta_i^q Y_i$ which means that (2) holds for $1 \le m \le \tau(j,k) - (q-1)$ and fails for $\tau(j,k) - q \le m \le \tau(j,k) - 1$; thus player i takes the second chore at dates $1 \le m \le \tau(j,k) - (q-1)$ and she waits for player j to become free at dates $\tau(j,k) - q \le m \le \tau(j,k) - 1$.

If the pure-on-role woa continuation prevails, taking the second task at any $t + m$ is better than waiting to play the pure-on-role SPE of the woa at $t + \tau(j,k)$ when

$$X_i \delta_i^m > \frac{1}{2}\left(\delta_i^{\tau(j,k)} Y_i\right) + \frac{1}{2}\left(\delta_i^{\tau(j,k)} X_i\right);$$

that is,

$$X_i\left(2 - \delta_i^{\tau(j,k)-m}\right) > \left(\delta_i^{\tau(j,k)-m} Y_i\right),$$

which holds whenever $X_i > (\delta_i/(2 - \delta_i))Y_i$, since

$$\frac{\delta_i}{2 - \delta_i} > \frac{\delta_i}{2 - \delta_i^{\tau(j,k)}} > \frac{\delta_i^{\tau(j,k)-m}}{2 - \delta_i^{\tau(j,k)-m}}.$$

In a mixed-on-role woa continuation which requires $X_i \le (\delta_i/(2 - \delta_i))Y_i$, player j, acting as #2, takes the remaining task with probability p_j. The value of p_j is such that player i, when acting as #2, is indifferent between delaying one period to attain expected gains $\delta_i Q_i$ or taking the task immediately and obtaining X_i. Thus player i's ex ante expected payoff is

$$Q_i = \frac{X_i}{\delta_i}.$$

With this expected gains in the woa continuation, player i's best reply is to take the remaining task at any date $t + m$ since

$$X_i \delta_i^m \ge \delta_i^{\tau(j,k)} Q_i = \delta_i^{\tau(j,k)}\left(\frac{X_i}{\delta_i}\right),$$

which is equivalent to $\delta_i^{m+1} \ge \delta_i^{\tau(j,k)}$, which always holds. (Note that at date $\tau(j,k) + t - 1$, player i is indifferent between taking the task and going on to the next period.)

If the i-mixed woa continuation prevails, again player i, when acting as #2, is indifferent between delaying one period to attain expected gains $\delta_i Q_i$ or taking the task immediately and obtaining X_i. Player i's ex ante expected payoff again is X_i/δ_i, and by the same reasoning as in the previous case, it is a best reply for player i to take the remaining task at any date $t+m$.

If the j-mixed woa continuation prevails, then in the continuation player i expects a payoff of X_i. At any date $t+m$, player i prefers taking the remaining chore now rather than waiting, $X_i > \delta_i X_i$. ■

Player i's best reply in subgames in which she alone is unoccupied depends on the woa continuation. In all woa continuation except the j-pure, if player i passes in each period until player j is again free, player i faces some probability of having to take the second task herself. In each period, it is a best reply for player i to concede. If the woa continuation is the j-pure, player j will take task h when task k has been completed. There are then two possibilities. When the game is $SP(j,k)$, player i is willing to bear the cost of the wait to enjoy the benefits associated with player j doing task h. It is a best reply for player i to pass in every period. Otherwise, player i is unwilling to wait $\tau(j,k)-1$ periods for player j to undertake the task, although player i would be willing to wait a shorter period of time. It is a best reply for player i to concede in $t+1$, but it is player i's best reply to pass at some later period and for all subsequent periods.

It follows directly from this analysis that in any SPE in which player j takes the first task at time t, player i takes the second chore at time $t+1$ unless the j-pure woa continuation prevails **and** $X(i,h) < \delta_i^{\tau(j,k)-1} Y(i,h)$. In this latter case, player j concedes on both tasks, and it is clear that player j concedes on task h immediately upon completing task k. Therefore, whoever takes the second task in equilibrium does so at her first opportunity. Equivalently, we can observe that it is only possible for both players to pass in a given period before the first concession is made.

These results are stated in Corollary 4 which fully characterizes SPE outcomes after the first task is taken.

COROLLARY 4. *Consider a SPE strategy profile. Suppose that j concedes first and takes task k at time t.* (A) *If the game is $SP(j,k)$ and the j-pure woa continuation prevails at all two-player subgames that follow j taking task k at time t, then j also concedes by taking task h at time $t+\tau(j,k)$.* (B) *Otherwise, player i concedes by taking task h at $t+1$.*

4.2. *Sufficient Conditions for a Unique Task-Sharing Outcome*

Before we address the complete characterization of SPEs, we provide conditions sufficient to guarantee that the SPE outcome is unique, and that this SPE outcome features task-sharing. The conditions given are simple and the proof of the result illustrates how Proposition 1 can be used in this two-chore context.

Under parameter values such that $\forall j$ and k the game is not $SP(j,k)$, an immediate consequence of Corollary 4 is that any SPE outcome features task-sharing and that the second task is taken without delay. Proposition 3 states that, in addition, there is no delay at the start of the game.

PROPOSITION 3. *Let* $n = 2$ *and* $c = 2$. *If condition* (1) *fails* $\forall i, j, h, k$, *then*: (A) *there is a unique SPE outcome*; *in this outcome, one task is started in each of periods* 0 *and* 1. (B) *Task-sharing without delay is surplus-maximizing*. (C) *If* #1 *always passes, if* $X(j,1) + \delta_j Y(j,2) \neq X(j,2) + \delta_j Y(j,1)$ *and* $Y(j,h) + \delta_j X(j,k) \neq X(j,k) + \delta_j Y(j,h)$ *for* $j, k, h = 1, 2$, $k \neq h$, *then the unique SPE outcome is Pareto optimal*.

Proof. Part (A). It is immediate from Corollary 4 that under condition

$$X(i,h) \geq \delta^{\tau(j,k)-1} Y(i,h) \qquad \forall i, j, k, h \tag{3}$$

in any SPE a first concession by one agent is followed immediately by a concession from her opponent. This is true regardless of which agent concedes first, and regardless of which task this agent chooses to undertake. We show that the first concession occurs in period 0. Let k_j be the task that agent j chooses if she concedes first:

$$X(j,k_j) + \delta_j Y(j,h_j) \geq X(j,h_j) + \delta_j Y(j,k_j)$$
$$j = 1, 2,\ k_j, h_j = 1, 2,\ h_j \neq k_j. \tag{4}$$

This implies that player j gets U_j if she concedes first and W_j if her opponent concedes first:

$$U_j = X(j,k_j) + \delta_j Y(j,h_j)$$
$$W_j = Y(j,k_i) + \delta_j X(j,h_i) \qquad j = 1, 2,\ k_i, h_i = 1, 2,\ h_i \neq k_i.$$

Since a first concession determines the outcome of the game, the players are in a war of attrition with payoffs given as above. By Corollary 1, to show that the first chore is undertaken in period 0 and that the SPE in the war of attrition is unique, it is sufficient to show that $U_i \geq \delta_i W_i$ for $i = 1$ and 2.

There are two cases to consider, depending on whether the two players would pick the same task on first concession. First suppose that $k_2 = h_1$, meaning that player 2 chooses task $k_2 : f$ she concedes first and player 1 chooses task $k_1 \neq k_2$. Then

$$U_1 = X(1,k_1) + \delta_1 Y(1,h_1) > \delta_1 Y(1,h_1) + \delta_1^2 X(1,k_1) = \delta_1 W_1$$

and

$$U_2 = X(2, h_1) + \delta_2 Y(2, k_1) > \delta_2 Y(2, k_1) + \delta_2^2 X(2, h_1) = \delta_2 W_2.$$

Second, suppose $k_2 = k_1$; then by Eq. (4)

$$U_1 = X(1, k_1) + \delta_1 Y(1, h_1) \geq X(1, h_1) + \delta_1 Y(1, k_1)$$
$$> \delta_1 Y(1, k_1) + \delta_1^2 X(1, h_1) = \delta_1 W_1$$

and

$$U_2 = X(2, k_1) + \delta_2 Y(2, h_1) \geq X(2, h_1) + \delta_2 Y(2, k_1)$$
$$> \delta_2 Y(2, k_1) + \delta_2^2 X(2, h_1) = \delta_2 W_2$$

as required.

Given a unique SPE in the war of attrition with payoffs $((U_1, W_2), (U_2, W_1))$, there is a unique outcome in the war of attrition. This means that there is a unique SPE outcome in the two-chore game. Given a permutation of the players, #1 either takes a first task in period 0 according to Eq. (4) in which case her opponent takes the second in period 1, or #1 passes, in which case #2 takes a first task in period 0 according to Eq. (4) and her opponent takes the remaining task in the next period.

Note, however, that the SPE strategies are not unique. For instance, if player j takes the first task in period 0 and player i fails to take the second task before i is free again at time $\tau(j, k)$, there are many possible continuations to the game, as given in Proposition 1.

Part (B). By contradiction. Suppose that (3) holds and that specialization is surplus-maximizing. Without loss of generality, assume that player 1 should undertake both tasks starting with task 1. This would imply that, in particular, player 1 doing both tasks in this order generates more surplus than player 2 undertaking task 2 after player 1 has started task 1,

$$X(1, 1) + Y(2, 1) + \delta_1^{\tau(1,1)} X(1, 2) + \delta_2^{\tau(1,1)} Y(2, 2)$$
$$> X(1, 1) + Y(2, 1) + \delta_1 Y(1, 2) + \delta_2 X(2, 2)$$

which implies

$$\delta_1^{\tau(1,1)} X(1, 2) + \delta_2^{\tau(1,1)} Y(2, 2) > \delta_1 Y(1, 2) + \delta_2 X(2, 2).$$

By (3) we have that

$$\delta_2 X(2, 2) \geq \delta_2^{\tau(1,1)} Y(2, 2).$$

Further, since $Y(1, 2) \geq X(1, 2)$,

$$\delta_1 Y(1, 2) \geq \delta_1 X(1, 2) > \delta_1^{\tau(1,1)} X(1, 2),$$

contradicting $\delta_1^{\tau(1,1)} X(1, 2) + \delta_2^{\tau(1,1)} Y(2, 2) > \delta_1 Y(1, 2) + \delta_2 X(2, 2)$.

Part (C). Without loss of generality, suppose that player 1 is #1 and player 1 passes. Let k_j be defined by Eq. (4). The equilibrium allocation has player 2 performing k_2 starting in period 0 and player 1 performing h_2 starting in period 1. We must show that this allocation cannot be Pareto dominated.

Player 2 chooses k_2 and thus

$$X(2, k_2) + \delta_2 Y(2, h_2) > X(2, h_2) + \delta_2 Y(2, k_2)$$

which implies that the equilibrium allocation is not Pareto dominated by one in which 2 starts with task h_2 and 1 follows with k_2. Player 1 chooses to pass as #1 and thus

$$\begin{aligned} Y(1, k_2) + \delta_1 X(1, h_2) &> X(1, k_1) + \delta_1 Y(1, h_1) \\ &\geq X(1, h_1) + \delta_1 Y(1, k_1) \end{aligned}$$

which implies that the equilibrium allocation is not Pareto dominated by one in which tasks are shared and 1 starts. Furthermore,

$$\begin{aligned} Y(1, k_2) + \delta_1 X(1, h_2) &> X(1, k_1) + \delta_1 Y(1, h_1) \\ &> X(1, k_1) + \delta_1^{\tau(1, k_1)} X(1, h_1) \end{aligned}$$

and

$$\begin{aligned} Y(1, k_2) + \delta_1 X(1, h_2) &> X(1, h_1) + \delta_1 Y(1, k_1) \\ &> X(1, h_1) + \delta_1^{\tau(1, h_1)} X(1, k_1) \end{aligned}$$

since $Y(1, k) > X(1, k)\ \forall k$ which implies that the equilibrium allocation is not Pareto dominated by player 1 doing both tasks. Further,

$$\begin{aligned} X(2, k_2) + \delta_2 Y(2, h_2) &> X(2, k_2) + \delta_2^{\tau(2, k_2)} X(2, h_2) \\ X(2, k_2) + \delta_2 Y(2, h_2) &> X(2, h_2) + \delta_2 Y(2, h_2) \\ &> X(2, h_2) + \delta_2^{\tau(2, h_2)} X(2, k_2). \end{aligned}$$

Thus the equilibrium allocation is not Pareto dominated by player 2 doing both tasks. ■

Under (3), once player j has started the first task, her opponent never wishes to wait for j to finish the first task, even if i were sure that j would then undertake the second task right away. This implies that a first concession by one player immediately triggers a second concession by the opponent. Since the outcome is determined once this first concession is made, the game is equivalent to a war of attrition in which players fight over who will be first to concede. Therefore, the SPE outcome of the game can be identified in two steps. In a first step, the task that each agent would choose were she to be the first to concede is identified yielding a vector of possible payoffs $((U_1, W_2), (W_1, U_2))$ where (U_j, W_i) represent the payoffs for players j and $i \neq j$ respectively when player j concedes first. In a second step, Proposition 1 can be used given $((U_1, W_2), (W_1, U_2))$ to determine how players behave at the start of the game. Proposition 1 describes all subgame perfect equilibria of the war of attrition, and thus it describes how players in the two-chore game act knowing that any first concession uniquely determines their payoffs. Under (3), postponing the first concession is never optimal since it simply delays the benefits from both tasks being done.

Condition (3) is also sufficient to guarantee that the surplus-maximizing outcome is task-sharing. Given the agents' preferences, the value of having the opponent perform both tasks is never worth the waiting time required for the second task to be performed. This result does not imply, however, that the unique SPE outcome necessarily yields a surplus-maximizing task assignment. The unique SPE outcome is task-sharing, and the surplus-maximizing outcome is task-sharing, but these two outcomes need not be the same. It is possible for players to share tasks but for each player to perform the "wrong" task from an efficiency standpoint. The example below provides a simple illustration of the incentives that create such a situation.

Proposition 3 also states that condition (3) along with some tie-breaking assumptions on payoffs are sufficient to ensure that the unique SPE outcome is Pareto optimal, as long as #1 chooses to pass in equilibrium. When (3) holds, passing as #1 can always be done with the assurance that #2 will concede in period 0. Consider for a moment the case of equivalent tasks. A player always passes as #1 since passing ensures that her opponent performs a task first, and her opponent performing a task yields a higher payoff than performing the task herself. In a general game in which tasks yield different benefits, a player concedes as #1 if the benefit from dictating the order in which tasks are performed outweighs the cost of postponing to period 1 the benefits from her opponent performing a task. If it is in a player's interest to concede as #1 then it is possible that having one player do both tasks in a different order effects a Pareto improvement over the SPE outcome. In the example below, a specialization outcome dominates the SPE outcome.

EXAMPLE 1. A unique SPE outcome is neither surplus-maximizing nor Pareto optimal.

Consider two players 1 and 2 and two tasks. Each task takes three periods to complete ($\tau = 3$) and players discount the future at a common rate of $\delta = 0.5$. Suppose that the benefits are

$$X(i,1) = 80 \quad X(i,2) = 400 \quad Y(i,1) = 200 \quad Y(1,2) = 1000 \qquad i = 1, 2.$$

It can be easily be checked that (3) holds.

Both players get larger benefits from task 2; indeed, performing task 2 oneself yields higher benefits than having one's opponent do task 1. The surplus-maximizing allocation is task-sharing: one player starts with task 2 in period 0 and her opponent undertakes task 1 in period 1. The high social benefits associated with performing task 2 are thus obtained at the earliest opportunity. The maximum total surplus is 1540 $(500 + 1040)$.

Notwithstanding this socially advantageous allocation, each player would rather have her opponent be the one to perform task 2. The high value from having the opponent perform task 2 outweighs the cost from not having task 2 performed immediately. The SPE is also task-sharing but each player chooses to perform task 1 first: $X(i,1) + \delta_i Y(i,2) = 80 + (0.5)(1000) = 580 > 500 = 400 + (0.5)(200) = X(i,2) + \delta_i Y(i,1)$. Furthermore, given that each player takes task 1 on first concession, each player prefers starting as #1 rather than passing and letting the other player concede first: $X(i,1) + \delta_i Y(i,2) = 580 > 400 = 200 + (0.5)(400) = Y(i,1) + \delta_i X(i,2)$. The total surplus generated with the players performing the "wrong tasks" in equilibrium is 980.

Each player's strategic incentive is to concede so as to force the opponent to perform task 2. The private incentives are not aligned with the social incentives in the sense that performing task 2 first yields greater social surplus. In particular, starting from the SPE outcome, the benefits accruing to both players can be increased by the player chosen #2 performing both tasks starting with task 2. The player chosen #1 gets $Y(i,2) + \delta_i^3 Y(i,1) = 1000 + (0.125)(200) = 1025 > 580$ while her opponent gets $X(j,2) + \delta_i^3 X(j,1) = 400 + (0.125)(80) = 410 > 400$. Thus, the SPE outcome is Pareto dominated by a specialization outcome. The total surplus under specialization is 1435.

When the game is APET $(Y(i,j,k) = Y,\ X(j,k) = X\ \forall j,k,i \neq j)$, each task generates the same social surplus regardless of who performs it. Players always pass as #1 because passing means getting the higher benefits from someone else undertaking a chore sooner and postponing the lower benefits from undertaking a chore oneself. The equivalent of (3) for the APET case is then sufficient to ensure that the SPE outcome features task-sharing without delay and that the SPE outcome is surplus-maximizing (and thus Pareto optimal).

COROLLARY 5. *Consider an APET game with* $n = 2$, $c = 2$. *If*

$$X \geq \delta^{\tau-1} Y \tag{5}$$

then there is a unique SPE outcome in which one task is started in each of periods 0 *and* 1. *The unique SPE outcome is surplus-maximizing.*

Proof. The proof follows straightforwardly from the proof of Proposition 3 after imposing the anonymity of players and the equivalence of tasks. ■

Condition (3) in the general game (and similarly condition (5) in the APET game) is sufficient for the SPE to feature task-sharing, but it is not necessary. When (3) fails, specialization becomes possible because player i, rather than doing the task herself, is willing to wait $\tau(j, h) - 1$ periods for j to become free and perform the second task. Yet specialization is not necessarily the equilibrium outcome because player j may not agree to take the second task upon becoming free—she may expect i to take it, or she may play a mixed strategy in the one-task continuation game. We explore these and other possible outcomes next.

4.3. *General Characterization of SPEs*

The procedure to characterize SPEs of the two-chore game used in the previous section can be readily extended to situations in which (3) does not hold—in other words, to games that are specialization feasible. Corollary 4 establishes that once a first concession is made by player j **and** once it is specified whether the woa continuation is the j-pure in the relevant subgames, then the SPE outcome of the game is uniquely determined. If the woa continuation is the j-pure and the game is $SP(j, k)$ then the second concession is made by player j at her earliest opportunity; otherwise the second concession is made by the opponent at her earliest opportunity. Thus the outcome of the game can be identified using the same two steps, first determining the task that each agent would choose were she to make the first concession, and then using Proposition 1 given the vector of payoffs $((U_1, W_2), (W_1, U_2))$ to determine the probability with which each player concedes first.

Consider the decision of agent j on which task to take, conditional on j being first to concede. As in the previous section, let k_j be the task that j would choose and let h_j be the other task. In the most general way, there are three possibilities. If agent j anticipates task-sharing to result from her concession regardless of which task she chooses to do first, then k_j maximizes j's payoff conditional on j taking the first task and i taking the second task immediately afterwards; k_j and h_j are defined by (4). If agent j anticipates specialization to result from her concession regardless of which task she chooses to do first, then k_j and h_j are such that

$$X(j, k_j) + \delta_j^{\tau(j, k_j)} X(j, h_j) \geq X(j, h_j) + \delta_j^{\tau(j, h_j)} X(j, k_j)$$
$$j, k_j, h_j = 1, 2, \quad h_j \neq k_j.$$

If j anticipates specialization if she starts with task k but she expects task-sharing if she starts with task $h \neq k$ then k_j is defined by

$$k_j = \begin{cases} h & \text{if } X(j,h) + \delta_j Y(j,k) \geq X(j,k) + \delta_j^{\tau(j,k)} X(j,h) \\ k & \text{if } X(j,k) + \delta_j^{\tau(j,k)} X(j,h) > X(j,h) + \delta_j Y(j,k) \end{cases} \tag{6}$$

and $j = 1, 2$, $h_j \neq k_j$, $k_j, h_j = 1, 2$.

The payoff vector $((U_1, W_2), (W_1, U_2))$ is fully determined by the possible outcomes (task-sharing or specialization) that follow a first concession. The outcome that follows player 1 (2) first conceding starting with task 1 and the outcome that follows player 1 (2) first conceding starting with task 2 together determine the task that 1 (2) would choose, k_1 (k_2). Given k_1 (k_2), the payoffs received by both players upon a first concession by 1 (2), (U_1, W_2) ((W_1, U_2)), are determined.

There are 10 distinct *payoff profiles*, i.e., 10 different types of payoff vectors, that could result from the game. These payoff profiles are categorized in Table I. The outcomes ("ts" for task-sharing and "sp" for specialization) that follow a first concession by players 1/2 when players 1/2 choose the payoff-maximizing task k_1/k_2 are given as columns of the table. The rows in Table I indicate the outcomes that ensue if players 1/2 choose the "other" task h_1/h_2. The entries of the table label the corresponding payoff profiles from 1 to 10. For example, consider payoff profile 3. In this profile, task-sharing prevails after a first concession when players choose their payoff-maximizing task. If players do not choose their profit-maximizing task, one player faces specialization after a first concession while the other can expect task-sharing. We consider the payoff profile to be of type 3 whether it is player 1 or player 2 who faces specialization by not choosing the payoff-maximizing task. Similarly, profiles 6 to 10 appear twice in the table, the first entry having player 1 in the role taken by player 2 in the second entry.

Given parameter values for the game ($X(j,k)$, $Y(j,k)$, $\tau(j,k)$, δ_j for $j = 1, 2$, $k = 1, 2$) generally only a subset of these possible profiles can be

TABLE I
Candidate SPE Payoff Profiles

Payoff profiles $h_1/h_2 \downarrow$	$k_1/k_2 \rightarrow$	ts/ts	sp/sp	sp/ts	ts/sp
ts/ts		1	4	7	7
sp/sp		2	5	8	8
sp/ts		3	6	9	10
ts/sp		3	6	10	9

sustained as SPE payoffs; conversely, each profile can be sustained for less than the full range of parameter values describing the preferences and abilities of the players. Before proceeding to the last step of the analysis that considers the probability with which each player is first to concede given these payoff profiles, we characterize the correspondence between possible game parameters and candidate SPE payoff profiles. This characterization uses the labelling of payoff profiles given in Table I, as well as two conditions on game parameters given presently.

DEFINITION 1. A game is **unavoidably specialized** for (j, k), denoted by $US(j, k)$, if and only if the game is $SP(j, k)$ and

$$X(j, h) \geq \delta_j Y(j, h) \qquad \text{for } h \neq k. \tag{7}$$

DEFINITION 2. A game is one in which **specialization can be chosen** for (j, k), denoted by $CS(j, k)$, if and only if the game is $SP(j, k)$, not $SP(j, h)$, and

$$X(j, k) > \delta_j Y(j, k) + \left[1 - \delta_j^{\tau(j,k)}\right] X(j, h). \tag{8}$$

Consider a game that is specialization feasible for j, k (i.e., $SP(j, k)$ holds). A game is, in addition, *unavoidably specialized* if by j conceding first starting with k, it is certain (by opposition to merely possible) that j takes both tasks. This requires that (a) the j-pure is the unique SPE of the one-task game to allocate the remaining task h and (b) player i can credibly threaten not to take the remaining task h. From Corollary 2, the j-pure is the unique SPE to allocate task h when (7) holds and when $X(i, h) < (\delta_i/(2 - \delta_i))Y(i, h)$. Player i can credibly threaten j not to take then remaining task h when the game is $SP(j, k)$. It is easy to show that $SP(j, k)$ implies $X(i, h) < (\delta_i/(2 - \delta_i))Y(i, h)$. Thus, in a game for which $SP(j, k)$ and (7) hold, if j concedes first starting with k then j must take both tasks. Note that games that are anonymous are never unavoidably specialized. Indeed, if a game is $US(j, k)$ then j is unwilling to wait even one period for i to perform task h but i is willing to wait at least $\tau - 1$ periods for j to perform this same task.

A specialization feasible game is one in which *specialization can be chosen* for j, k if player j prefers to specialize starting with task k than to task-share starting with task h. This requires that player j prefers the specialization outcome and that the game is not $US(j, h)$ so that task-sharing would be feasible if j undertook task h first. From (6), player j chooses a specialization outcome starting with k when (8) holds. From (7), the game is not $US(j, h)$ either when

$$X(j, k) < \delta_j Y(j, k) \tag{9}$$

or when the game is not $SP(j, h)$; namely, $X(i, k) \geq \delta_i^{\tau(j,h)-1} Y(i, k)$. Since (9) contradicts (8), it must be that the game is not $SP(j, h)$. Note that games with equivalent tasks are never CS games. Indeed, in CS games player j is willing to perform both tasks herself so as to force tasks to be performed in a particular order.

In the first column of Table I, task-sharing is the equilibrium outcome regardless of which player concedes first. It has already been established that (3), which holds when the game is not specialization feasible, is a sufficient condition for profile 1 to be the unique payoff profile (Proposition 3). Under profiles 2 and 3, the game is specialization feasible but a task-sharing outcome obtains. A task-sharing payoff profile in the first column can be supported as a SPE payoff **unless** task-sharing is *not feasible* following a player first conceding, or unless specialization is always *chosen* by a player when she concedes first. The following lemma states the result formally in terms of the conditions defined above.

LEMMA 2 (Feasibility of Task-Sharing Outcomes). *Profiles* 1, 2, *and* 3 *can be supported as SPE payoffs unless, for some* k, $j = 1$ *or* 2 *and for* $h \neq k$, *the game is*:

(A) $US(j, k)$ *and* $US(j, h)$ *or*

(B) $US(j, k)$ *and* $CS(j, k)$.

Proof. The proof that conditions (A) or (B) implies that a task-sharing outcome is not possible is included in the text below. If neither condition (A) nor condition (B) is satisfied, then (1) for at least one task (say task 1) there exists a SPE in the one-task game that is not the j-pure or the i-pure, and (2) if the SPE in the one-task game for task 2 is unique and it is the j-pure (i-pure), then player j (i) would choose task 1 if she were to concede first. Therefore, a task-sharing SPE outcome can always be constructed by choosing the woa continuation not to be the pure on identity whenever possible. ■

Task-sharing is *not feasible* whenever j conceding unavoidably leads to specialization, regardless of the task that j would choose to perform. Specialization is unavoidable for j starting with task k whenever the game is $US(j, k)$. Therefore, if the game is both $US(j, k)$ and $US(j, h)$, then specialization must follow a first concession by j. A task-sharing outcome can similarly ruled out if *specialization is unavoidable* when j starts with k, and if specialization is actually chosen by j over the task-sharing outcome that would ensue were j to perform task h first. From the conditions above, the game being $US(j, k)$ guarantees that j undertakes both tasks if she starts with task k. When, in addition, the game is $CS(j, k)$, then player j chooses this specialization outcome over the alternative of task-sharing but changing the order of the tasks. Under both of these circumstances, a task-sharing payoff profile in the first column of Table I is impossible.

In the last three columns of Table I, specialization is a possible outcome. Whether specialization or task-sharing results depends on the identity of the player making the first concession. In profiles 4, 6, 7, and 10, one player chooses a task that leads to specialization when task-sharing is feasible which requires that the game be $CS(j,k)$ for some (j,k). As mentioned earlier, the restrictions on the possible parameters imposed by such games are severe and, in particular, APET games are excluded. Profile 6, in which one agent (say i) always faces specialization while the other (say j) chooses specialization, is impossible. To support this payoff profile the game must be $SP(i,h)$, $SP(i,k)$, and $CS(j,k)$. The game must be $CS(j,k)$ for j to choose specialization and this implies $X(j,k) \geq \delta_j Y(j,k)$. Player j prefers doing task k herself rather than waiting one period for her opponent to perform this task. Player j must then be unwilling to wait $\tau(i,h)-1$ periods for her opponent to do task k after i has performed task h. Therefore the game cannot $SP(i,h)$.

A simple characterization of the game parameters for which each payoff profile can be supported in a SPE is given in the next proposition. The proposition uses the profile labelling of Table I (with the last column omitted as the conditions can be obtained by reversing the roles of the two players in other cells). We use $\sim$ to mean "not." For example, Table II says that profile 2 can be sustained as a SPE payoff whenever, for $j=1,2$, the game is $SP(j,h_j)$, not $US(j,k_j)$ and not $CS(j,h_j)$.

PROPOSITION 4. *A profile of Table I can be sustained as a SPE payoff iff the conditions given in the corresponding cell of Table II hold.*

Proof. Whenever $US(j,k)$ fails for all (j,k), either (1) it is possible to specify the woa continuation not to be the pure on identity or (2) if the unique SPE of the one-task game is the j-pure then i cannot credibly threaten to wait for j to perform the remaining task. Thus, it is possible to construct a strategy profile such that each player expects task-sharing to follow a first concession regardless of the task chosen first. Profile 1 can be sustained as a SPE. Conversely, if Profile 1 is supported by a SPE, then it cannot be the case that the game is $US(j,k)$ for some (j,k). Indeed, if the game were $US(j,k)$ then j starting with k yields unavoidably to specialization which is precluded by profile 1. It is similarly routine to check that the conditions given for other profiles guarantee that the profiles can be supported in a SPE. Profile 6 is impossible as a SPE. ∎

The last step of the analysis considers the probability with which each player is first to concede given a payoff vector $((U_1,W_2),(W_1,U_2))$. In the

TABLE II
Conditions to Sustain Payoff Profiles in a SPE

Conditions $h_1/h_2 \downarrow$	$k_1/k_2 \rightarrow$	ts/ts	sp/sp	sp/ts
ts/ts		$\sim US(j,k)$ $\forall j,k$	$CS(1,k_1)$ $CS(2,k_2)$ $k_1=k_2$	$CS(1,k_1)$ $\sim US(2,k)\ \forall k$
sp/sp		$SP(j,h_j)\ \forall j$ $\sim US(j,k_j)\ \forall j$ $\sim CS(j,h_j)\ \forall j$	$SP(j,k)$ $\forall j,k$	$SP(1,k)\ \forall k$ $SP(2,h_2)$ $\sim CS(2,h_2)$ $\sim US(2,k_2)$
sp/ts		$SP(1,h_1)$ $\sim US(j,k_j)\ \forall j$ $\sim CS(1,h_1)$	impossible	$SP(1,k)\ \forall k$ $\sim US(2,k)\ \forall k$
ts/sp		$SP(2,h_2)$ $\sim US(j,k_j)\ \forall j$ $\sim CS(2,h_2)$	impossible	$CS(1,k_1)$ $\sim CS(2,h_2)$ $SP(2,h_2)$

following proposition, we use Proposition 1 to consider the behavior at the start of the game for task-sharing SPE outcomes.

PROPOSITION 5. *Consider games that admit task-sharing outcomes* (*Lemma* 2). (A) *If there does not exist* (j,k) *such that the game is* $US(j,k)$, *then a task-sharing SPE in which the first concession occurs in period* 0 *always exists*. (B) *If tasks are equivalent or if the allocation of tasks to players is independent of who concedes first* $(k_2 = h_1)$, *in any task-sharing SPE the actions taken in period* 0 *by each player are specified uniquely and they yield immediate concession*.

Proof. Part (A). Consider all one-task subgames. Set the woa continuations in these subgames to be anything that is a SPE in the one-task subgame, but that is not a pure on identity SPE. Given that the game is such that there does not exist (j,k) such that the game is $US(j,k)$, this is always possible. From Lemma 1, this implies that once a first task is taken, task-sharing results. The relevant payoff vector is then

$$U_1 = X(1,k_1) + \delta_1 Y(1,h_1) \qquad W_2 = Y(2,k_1) + \delta_2 X(2,h_1)$$
$$W_1 = Y(1,k_2) + \delta_1 X(1,h_2) \qquad U_2 = X(2,k_2) + \delta_2 Y(2,h_2).$$

If $k_2 = h_1$, then

$$U_1 = X(1,k_1) + \delta_1 Y(1,h_1) > \delta_1 Y(1,h_1) + \delta_1^2 X(1,k_1) = \delta_1 W_1$$
$$U_2 = X(2,h_1) + \delta_2 Y(2,k_1) > \delta_2 Y(2,k_1) + \delta_2^2 X(2,h_1) = \delta_2 W_2.$$

If $k_2 = k_1$, then, by definition of k_i, h_i

$$\begin{aligned} U_1 &= X(1, k_1) + \delta_1 Y(1, h_1) \geq X(1, h_1) + \delta_1 Y(1, k_1) \\ &> \delta_1 Y(1, k_1) + \delta_1^2 X(1, h_1) = \delta_1 W_1 \end{aligned}$$

and

$$\begin{aligned} U_2 &= X(2, k_1) + \delta_2 Y(2, h_1) \geq X(2, h_1) + \delta_2 Y(2, k_1) \\ &> \delta_2 Y(2, k_1) + \delta_2^2 X(2, h_1) = \delta_2 W_2. \end{aligned}$$

From Proposition 1, if $U_j \geq \delta_j W_j$ for $j = 1$ or 2 then there is a concession in period 0.

Part (B). Tasks are equivalent whenever $X(j, h) = X(j)$; $Y(j, k) = Y(j)$. Since

$$\begin{aligned} U_j &= X(j) + \delta_j Y(j) > \delta_j Y(j) + \delta_j^2 X(j) = \delta_j W_j \qquad j = 1, 2 \\ U_j &= X(j) + \delta_j Y(j) < Y(j) + \delta_j X(j) = W_j \qquad j = 1, 2 \\ &\Leftrightarrow (1 - \delta_j) X(j) < (1 - \delta_j) Y(j) \qquad j = 1, 2. \end{aligned}$$

we have $\delta_j W_j < U_j < W_j$ for $j = 1$ and 2. From Lemma 1, in period 0, each player passes as #1 and concedes as #2. The actions at the beginning of the game are unique and they lead to immediate concession.

The allocation of tasks to players is independent of who concedes first whenever $k_2 = h_1$. As shown in Part (A), this implies that $U_j > \delta_j W_j$ and the first concession occurs in period zero. Moreover, one of the following three payoff rankings holds: $\delta_j W_j < U_j < W_j$ $j = 1, 2$; or $U_j > W_j$ and $U_i < W_i$, $i \neq j$; or $U_j > W_j$, $j = 1, 2$. From Lemma 1, for each of these three possibilities the actions at the beginning of the game are uniquely determined. ■

If a game admits profile 1 in a SPE, given that each agent performs exactly one task on and off of the equilibrium path, delaying to have the opponent move first does not pay and the first concession occurs in period 0. This is the result of Part (A). With profiles 2 and 3, Proposition 1 specifies a variety of equilibria; a SPE in which concession is immediate may coexist with a delayed task-sharing equilibrium. In particular, it is possible that players disagree on who should perform which task and that the players will randomize at the start of the game to determine who concedes first. However, if tasks are equivalent or if the allocation of tasks to players is independent of who moves first, then again concession is immediate.

A SPE that features specialization by both players is consistent with exactly two sets of restrictions on game parameters, corresponding to profiles 4 and 5. In the first circumstance, both players choose to specialize starting with task k even though a first concession on task h would trigger task-sharing, requiring that the game be $CS(1, k)$ and $CS(2, k)$. These requirements are very restrictive;[10] in particular, the equivalence of tasks is precluded since players bear the cost of specialization in order to effect a certain task order. For the ensuing payoff vector $((U_1, W_2), (W_1, U_2))$, it can only be shown that $U_i < W_i$. This implies that players do not necessarily concede as #1, but it does not otherwise constrain the behavior of players at the start of the game. Any of the actions implied by Proposition 1 are possible so that the first concession may be immediate or may feature delay. In the second circumstance, both players specialize regardless of the task chosen first, which requires that the game be $SP(j, k)$ for all (j, k). (Note that these two sets of circumstances are mutually exclusive since a game that is $CS(j, k)$ is not $SP(i, h)$.) As the following lemma states, when players agree on the task that should be done first, then there are three possible starts to the game. For players to agree on the task to be done first, it is sufficient (but not necessary) that tasks be equivalent or that players be anonymous.

LEMMA 3. *Consider SPE strategies in which players specialize regardless of the task chosen upon first concession. Suppose that* $k_1 = k_2$. *Then there are three possible outcomes*: i *concedes first in period* 0; j *concedes first in period* 0; *in each period* t, #1 *passes and* #2 *concedes first with a given player-specific probability* $p_i \in (0, 1)$, $i = 1, 2$.

Proof. The result follows immediately from Proposition 1 and $U_i < (\delta_i/(2 - \delta_i))W_i$. ■

In SPEs not yet discussed, the outcomes are j-specialized. Specialization is the equilibrium outcome if j concedes first while task-sharing results if $i \neq j$ makes the first concession. In these SPEs, no general restrictions can be obtained on the set of payoff vectors $((U_1, W_2), (W_1, U_2))$; Proposition 1 must be applied on a case by case basis to determine possible behavior at the start of the game.

It is clear from the preceding discussion that imposing either the equivalence of tasks or the anonymity of players softens the conflict over the first concession and leads to sharper predictions regarding the outcome at the beginning of the game. Imposing the equivalence of tasks means that the game cannot be CS and thus eliminates outcomes in which a player chooses specialization over task-sharing; imposing the anonymity of players eliminates game that are US so that task-sharing outcomes can always be supported as SPE outcomes. We summarize our findings regarding SPE outcomes for the case of APET games in the following proposition.

[10]It is tedious but easy to check that a game that is $CS(j, k)$ and $CS(i, k)$ is not $SP(i, h)$ and not $SP(j, h)$.

PROPOSITION 6. *For an APET game*: (A) *Immediate task-sharing is always an SPE outcome and any task-sharing SPE yields immediate task-sharing*. (B) *If* $X < \delta^{\tau-1}Y$, *specialization SPEs and j-specialized SPEs exist*. (C) *In any specialization SPE*, *there are three possibilities*: *i concedes first at* 0, *or j concedes first at* 0, *or at each* t, #1 *passes and* #2 *concedes first with a given player-specific probability* $p_i \in (0, 1)$. (D) *In any j-specialized SPE*, *there are four possibilities*: *i concedes first at* 0; *j concedes first at* 0; *at each* t, #1 *passes and* #2 *concedes first with a given player-specific probability* $\tilde{p}_i \in (0, 1)$; *at each* t, *i passes as* #1 *and concedes first as* #2 *with a given player-specific probability* $q_i \in (0, 1)$, *while j concedes first as* #2 *with probability* 1 *and concedes first as* #1 *with a given probability* $r_j \in (0, 1)$.

Proof. Parts (A), (B), and (C) are corollaries of Proposition 5 and Lemma 3.

Part D. If the game is APET and there is a j-specialization equilibrium, then the payoff vector must correspond to either profiles 8 or profile 9. It is sufficient to show that the payoff vectors in these cases are such that $U_j < (\delta_j/(2-\delta_j))W_j$ and $U_i < W_i$ when tasks are equivalent and players are anonymous. We have

$$\begin{aligned} U_i &= X + \delta Y < \delta^{\tau-1}Y + \delta Y \\ &< Y + \delta^{\tau} Y = W_i \\ U_j &= X + \delta^{\tau} X \\ &< \frac{\delta}{2-\delta}Y + \frac{\delta^2}{2-\delta}X = \frac{\delta}{2-\delta}W_j \end{aligned}$$

since $X < \delta^{\tau-1}Y$ implies $X < \frac{\delta}{2-\delta}Y$ and since $\delta^{\tau}X < (\delta^2/(2-\delta))X$ because $\delta^{\tau}(1-\delta) < \delta^2(1-\delta^{\tau-2})$. ■

Our main findings on the structure of the SPE for general (2×2) games can now be summarized. After a first concession, the outcome of the game is uniquely determined given parameters of the game that describe the tastes and abilities of agents, and given whether the woa continuation is the pure on identity. The second concession is always immediate and delay only occurs at the beginning of the game. The game at period zero has the structure of a classic war of attrition; correspondingly, a wide range of behavior can generally be observed on the first concession. The range of payoff profiles that can be supported in a SPE depends on whether specialization is feasible, unavoidable, or preferred. Undelayed task-sharing SPEs exist as long as (players and tasks) differences are not too large. In APET games, any task-sharing equilibrium involves immediate concession.

5. TWO-PLAYER APET GAMES WITH MANY CHORES

In an APET game with two players and $c \geq 2$ chores, the surplus-maximizing allocation features task-sharing. An individual volunteers for the first task in period 0, her opponent volunteers in the next period for a second task, and each individual takes on a new task the moment she finishes the last. When the number of tasks is even, each player performs $\frac{c}{2}$ tasks; when the number of tasks is odd, the player volunteering first gets the "extra" task and performs $\frac{c+1}{2}$ tasks while her opponent gets away with doing $\frac{c-1}{2}$ tasks.

In this section, we show that regardless of the number of tasks, and regardless of how large the benefits are when the opponent carries out a task, there always exists a SPE that supports the surplus-maximizing outcome. We present sufficient conditions under which the SPE is unique and surplus-maximizing. We also present much weaker conditions that ensure that the surplus-maximizing SPE outcome is unique among a set of SPEs in symmetric profiles. Whenever the surplus-maximizing outcome is unique, we can trust that the voluntary private provision yields a fair and efficient allocation. The equivalence of tasks, the anonymity of players combined with symmetry of behavior constitutes one set of factors that ensure that this is the case.

5.1. *Preliminary Results*

We present two preliminary results that are used to establish the main propositions on existence and uniqueness.

LEMMA 4. *Let $n = 2$ and $c \geq 2$. Consider a strategy profile that yields task-sharing and surplus maximization. If c is even, or if c is odd and $X \geq \frac{\delta}{2-\delta} Y$, then it is a best reply for #2 to take the first task at her first opportunity when both players are free. If c is odd and $X < \frac{\delta}{2-\delta} Y$, then $\exists\ c^* < \infty$ such that, when both players are free, it is a best reply for #2 to take the first task at her first opportunity whenever $c \geq c^*$.*

Proof. See the Appendix. ■

The above result states that, if the number of tasks is even, then a player chosen second in the permutation always takes a first task at her first opportunity when she expects task-sharing. Recall that in the two-chore context, if a player is willing to take a second chore when her opponent is the first mover, then she has no reason to pass on taking the first task if she is given the chance. Indeed, she will be doing exactly one chore whether she starts work or whether her opponent does. The same argument can be made in the general c-chore case. Anticipating that she will perform $c/2$ tasks and so will her opponent, a player might as well start work at the first opportunity.

Refusing the first task does not change the number of tasks she performs; it only postpones the benefits from all tasks.

The case of an odd number of tasks is less straightforward. Here, there is a cost to being the first one to volunteer in a task-sharing arrangement, namely having to take on the "extra" task. A player chosen #2 will nevertheless volunteer whenever

$$\begin{aligned} &X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + Y\delta^{2\tau+1} + \cdots + X\delta^{\frac{c-1}{2}\tau} + Y\delta^{\frac{c-1}{2}\tau+1} + X\delta^{(\frac{c-1}{2}+1)\tau} \\ &\geq \frac{\delta}{2}\Big(X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + Y\delta^{2\tau+1} + \cdots + X\delta^{\frac{c-1}{2}\tau} + Y\delta^{\frac{c-1}{2}\tau+1} \\ &\quad + X\delta^{(\frac{c-1}{2}+1)\tau}\Big) + \frac{\delta}{2}\Big(Y\delta^{\tau} + X\delta^{\tau+1} + Y\delta^{2\tau} + X\delta^{2\tau+1} + \cdots + Y\delta^{\frac{c-1}{2}\tau} \\ &\quad + X\delta^{\frac{c-1}{2}\tau+1} + Y\delta^{(\frac{c-1}{2}+1)\tau}\Big) \end{aligned} \tag{10}$$

which is equivalent to

$$\frac{1-\delta^{\frac{c-3}{2}\tau}}{1-\delta^{\tau}}\big[2(X+Y\delta) - \delta(1+\delta)(X+Y)\big] \geq \delta^{(\frac{c-1}{2})\tau}\big[\delta Y - (2-\delta)X\big].$$

If $X \geq \frac{\delta}{2-\delta}Y$ then the preceding equation holds since the left-hand side is positive, and thus (10) holds. If the value of her opponent doing a task is relatively low compared to the benefits of doing it herself, then it is less costly to start the task-sharing arrangement than to postpone the benefits from all tasks. Similarly, if the number of tasks is large ($c > c^*$), then volunteering for the first task is a best reply because the benefits of having the opponent do the extra task, discounted far into the future, are low compared to the cost of postponing the benefits from all tasks.

This result has an immediate practical implication. If a job can be subdivided into tasks of various lengths, then it is easier to get players to start on a task-sharing arrangement if the work is divided into many "small" tasks. With an even number of chores, subdividing tasks does not matter; with an odd number of chores, having small tasks minimizes the importance of the "extra" task and thus minimizes the incentive for players to stall that the "extra" task creates. This is the essence of the following corollary.

COROLLARY 6. *Suppose that each task of length τ can be divided into μ subtasks, each taking τ/μ periods to complete, and each yielding payoffs to i of X/μ if i does a subtask and of Y/μ if j performs a subtask. Consider a strategy profile in the $c\mu$-task game where surplus is maximized. There always exists a division into subtasks such that it is a best reply for the player who is #2 to take the first subtask at his first opportunity.*

Proof. See the Appendix. ■

5.2. *Existence*

Lemma 4 is used directly to prove the general existence result of the following proposition.

PROPOSITION 7. *In the c-task, two-player game, a SPE that supports the surplus-maximizing outcome exists.*

Proof. See the Appendix. ■

If players are willing to start the game by taking a task right away regardless of the number of chores, then it implies that in a c-chore game, players are willing to take a task in every subgame. There is no delay. When the value of performing the chore oneself is high enough, then in a subgame with κ chores remaining, the promise of future task-sharing is sufficient to induce a player to take the next task. When the value of the opponent performing the task is high enough, then the equilibrium specifies that the player who has most recently finished a task will always abstain from taking another task, forcing the player who should take the next task to do so.

5.3. *Uniqueness*

The following result provides sufficient conditions under which the *unique* SPE outcome of our game is surplus-maximizing. With an even number of chores, the condition follows directly from the condition given in the 2×2 APET game. When the number of chores is odd, the sufficient condition required is much more stringent.

PROPOSITION 8. *Consider APET games with $n = 2$. If $X \geq \delta^{\tau-1}Y$ and c is even, or if $X \geq \delta Y$ and c is odd, then there is a unique SPE outcome and that outcome is surplus-maximizing.*

Proof. Let c be even. Suppose that, in period t, player i must decide whether to take a task (either because both players are free and she is #2, or because player j is busy). If player i turns down the task, then either player i does this task later, or player i transfers the task to player j. If player i does the task later, turning down the task now just postpones the benefits from the task, and this cannot be a best reply. If the task is transferred to player j, then instead of each player performing $c/2$ tasks as is the case along the equilibrium path, player i performs $c/2 - 1$ tasks and is busy for $(c/2 - 1)\tau$ periods, while player j performs $c/2 + 1$ tasks and is busy for $(c/2 + 1)\tau$ periods. This implies immediately that player i must be idle for τ periods while player j is busy. The costs from being idle is the benefit foregone from the task not being done. This cost is minimized when the benefits are discounted most heavily, i.e., when player i declines to her last task. This minimum cost is then $X\delta^{(c/2)\tau}$ if player i performs the first task and $X\delta^{(c/2)\tau+1}$ if player i performs the second task. The maximum benefits from being idle are the benefits associated with player j performing the $c/2+1$st task with no delay. These benefits are $Y\delta^{(c/2+1)\tau+1}$ if player j also performed the second task and $Y\delta^{(c/2)+1)\tau}$ if player j performed the first task. The condition $X \geq \delta^{\tau-1}Y$ ensures that $X\delta^{(c/2)\tau} \geq Y\delta^{(c/2+1)\tau+1}$ and that $X\delta^{(c/2)\tau+1} \geq Y\delta^{(c/2+1)\tau}$.

Let c be odd. The only possible $\kappa = 1$ continuation is the pure SPE on roles. Without loss of generality suppose that player i would have started and done $\frac{c+1}{2}$ chores, while j would have taken the second chore and performed $\frac{c-1}{2}$ chores in total. It is sufficient that player i is not willing to delay in the least costly way hoping that the $\frac{c+1}{2}$st chore will be reassigned to player j. The least costly way to delay is to delay on the task farthest into the future, which means that i should wait for j to be free for the $\frac{c+1}{2}$st task to be assigned. For i not to be willing to do this, the cost must be higher than the gain; i.e., $X\delta^{\tau} \geq \frac{1}{2}Y\delta^{\tau+1} + \frac{1}{2}X\delta^{\tau+1}$. The condition $X \geq \delta Y$ is sufficient. ■

In a surplus-maximizing SPE, it is never in the interest of an agent to wait for her opponent to perform one task more than is prescribed by a task-sharing arrangement. The value of performing the task oneself is high enough so that the cost of being idle while waiting for the opponent (the benefits foregone from not doing the chore oneself, X) exceed the benefits from having the opponent perform the task at a later date. When c is even, for example when $c = 2$, if the opponent has just started the first chore, the cost of being idle is $\delta^{\tau-1}Y$. When c is odd, for example when $c = 3$, if the opponent is finishing the second chore, the cost of being idle is δY (the player under consideration finishes the first chore at τ while the opponent finishes the second chore at $\tau + 1$).

Next, we restrict our attention to symmetric strategy profiles. We will show that under mild conditions, the surplus-maximizing SPE outcome is unique in the class of SSPEs.

A surplus-maximizing SPE always features task-sharing, but a task-sharing SPE may or may not be an SSPE. A strategy profile is *not* symmetric when strategies depend on the identity of the player rather than merely on her role as first or second in the permutation. For instance, strategies specifying that player i always takes odd-numbered tasks and always passes on even-numbered tasks while player j always passes on odd-numbered tasks and always takes even-numbered tasks yield a surplus-maximizing task-sharing arrangement, but the strategy profile is not symmetric. A SSPE that is surplus-maximizing features undelayed task-sharing and, necessarily, #2 starts work in period 0.

For the remainder of this section, we make the following assumption.

Assumption. In subgames with only one agent free, if she is indifferent between passing and taking a task, she takes a task.

Suppose that agent i alone is free one period before a subgame in which both players are free and the woa continuation is either the mixed on roles or the j-mixed. Player i's payoff from taking the task now is the same as her payoff from waiting one period for j to become free. The assumption breaks the tie in favor of volunteering.

The next lemma is a stepping stone for the results to follow.

LEMMA 5. *Let c be even. Consider a SPE strategy profile that is symmetric for all subgames in which $c-1$ chores are left and both players are unoccupied. Then the SPE is a surplus-maximizing SSPE.*

Proof. See the Appendix. ■

The result establishes that for a game with an even number of chores, symmetric behavior[11] in the continuation game implies surplus maximization. In the two-chore game, for instance, equilibrium behavior in the $c - 1 = 1$-chore subgame is symmetric when agents play the pure strategy equilibrium on roles or a symmetric mixed strategy equilibrium. With either continuation, if a player has taken the first chore, her opponent has the incentive to take the second task immediately. Given that her opponent takes the second chore at her first opportunity, and given that by Lemma 4 #2 in period 0 always starts when she expects task-sharing, the SPE in the two-chore game is symmetric and surplus-maximizing. Similarly, in a general c-chore game, if behavior is symmetric in the $c-1$ continuation and if a player has taken the first chore, then her opponent can never do better by waiting for her to become free again, and so the opponent prefers to take the second task immediately. Given that the player chosen #2 in period 0 can expect task-sharing, then by Lemma 4 she takes a task and gets the surplus-maximizing task-sharing arrangement going immediately. Symmetric behavior in the continuation implies that players can never do better than the task-sharing arrangement, and thus that there is no gain from waiting for the other to volunteer.

Lemma 5 can be used to obtain a similar result for games with an odd number of chores.

LEMMA 6. *Let c be odd and assume that* (10) *holds for c. Consider a SPE strategy profile that is symmetric for all subgames in which $c-2$ chores are left and both players are unoccupied. Then the SPE is a surplus-maximizing SSPE.*

[11]The result is proven by induction. Since the lemma holds for any even number of chores, the behavior of the players is symmetric in every subgame.

Proof. Follows directly from the previous result. ■

If the strategies in the $c-2$ chore subgames with both players available are symmetric, then by Lemma 5, the SPE in the $c-1$ chore subgame is the surplus-maximizing SSPE. If the first task has been taken, then the other player can do no better than to take the second task at her first opportunity since she will be sharing tasks if she waits for the $c-1$ chore continuation. For the SPE of the c-chore game to be the surplus-maximizing SSPE, the only additional requirement is for the player chosen #2 in period 0 to be willing to take the first task. She will be willing to do so when (10) holds for c. Note that this condition is very mild, as (10) always eventually holds as the number of chores increases.

The previous lemmas immediately yield the following result.

PROPOSITION 9. *If* (10) *holds for* c *when* c *is odd*, *or at* $c-1$ *when* c *is even*, *then there is a unique SSPE and it is surplus-maximizing.*

Proof. Follows directly from the previous result. ■

In contrast to the message delivered by models with one public good, symmetric behavior in a multitask environment goes hand in hand with surplus maximization. With one public good, symmetric behavior can only mean that both agents wait for the other to volunteer; with many public goods, symmetric behavior can mean that each agent is willing to do her share of the work if her opponent does the same. Further, as for the previous lemma, the condition for Proposition 9 to hold includes simply having a large number of chores. In this sense, under symmetric behavior, increasing the number of tasks promotes efficiency.

6. EXTENSIONS TO MORE THAN TWO PLAYERS

Even though our results focus on the case of two players, our model, as formulated in Section 2, can accommodate any number of players and chores. The number of players is only important relative to the number of chores to be performed. A situation where the number of players exceeds the number of chores ($n > c$) is likely to yield the kind of conflicts observed in the usual war of attrition.[12] Each individual is tempted to wait for another to volunteer since in equilibrium, some players will get away with doing nothing while others do the work. Symmetric behavior would yield delays

[12] Ponsatí and Sákovics (1996), as well as Bulow and Klemperer (1999), examine such a generalized war of attrition. Although their context and concerns are very different from ours, their results do indicate that the situation of more "players" than "prize" mirrors most of the features of the usual war of attrition.

before the chores are done, while asymmetric equilibria where designated players volunteer would ensure that there is no delay. Even without delay, efficiency may be hampered by the fact that, given their talents, players and tasks are mismatched.

In contrast, a situation where the number of chores is greater than the number of players and where $\tau > n$ is likely to feature task-sharing much in the manner of the two-player, c-chore game. When $\tau > n$, all players can be busy at once: while one task (say task k) is being performed, all idle players have an opportunity to take one of the remaining chores before task k is finished. Surplus maximization in symmetric games of this sort features task-sharing. As in the results presented for the two-player case, we expect that the promise of others performing the remaining tasks serves as the incentive to volunteer. As an illustration of a generalization in this direction, consider the following result.

PROPOSITION 10. *Consider an APET game with $n = c$ and $\tau > n$. If $X \geq Y\delta^{\tau-(n-1)}$, then there is a unique SPE and it is surplus-maximizing.*

Proof. See the Appendix. ■

This proposition generalizes Corollary 5. As long as each player prefers to take on a chore herself rather than wait for another player to become free to perform it (where $\tau - n - 1$ is the shortest wait) then there is no delay. The SPE features surplus-maximizing task-sharing.

7. CONCLUSION

We have studied a noncooperative game of task-sharing that models situations where individual members of a group must voluntarily undertake chores that yield collective benefits and impose private costs to the individual who carries them out. Our contribution is an exploration of the fundamental differences between the situation in which only one chore is at stake—a situation that can be modeled as a war of attrition—and situations in which several chores must be completed. Several factors work in favor of efficiency and fairness so that multiple task-sharing conflicts may actually be resolved, in spite of the noncooperative behavior of individuals, in a smoother, fairer, and more efficient fashion than single-task-sharing problems. Symmetric behavior, which in a single-task context promotes confrontation and inefficiency, is one of the factors that promotes fairness and efficiency in a multiple-task context. When symmetric strategies are allied to the anonymity of players and the equivalence of tasks, these factors promote equilibrium outcomes with quick, evenly distributed, and nonconfrontational allocations of chores. Having refrained throughout from commenting on the implications of our work for gender equality, we hope that the reader will forgive us if we now confess that we find this message pleasantly reassuring.

APPENDIX

Proof of Lemma 4. Suppose that c is even. Given a surplus-maximizing task-sharing arrangement, the player who is #2 takes the first task at the earliest opportunity whenever

$$\begin{aligned} &X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + Y\delta^{2\tau+1} + \cdots + X\delta^{\frac{c}{2}\tau} + Y\delta^{\frac{c}{2}\tau+1} \\ &\geq \frac{\delta}{2}\left(X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + Y\delta^{2\tau+1} + \cdots + X\delta^{\frac{c}{2}\tau} + Y\delta^{\frac{c}{2}\tau+1}\right) \\ &\quad + \frac{\delta}{2}\left(Y\delta^{\tau} + X\delta^{\tau+1} + Y\delta^{2\tau} + X\delta^{2\tau+1} + \cdots + Y\delta^{\frac{c}{2}\tau} + X\delta^{\frac{c}{2}\tau+1}\right) \qquad (11) \end{aligned}$$

which holds since

$$(X + \delta Y)\sum_{h=0}^{\frac{c}{2}} \delta^{h\tau} \geq \delta(X + \delta Y)\sum_{h=0}^{\frac{c}{2}} \delta^{h\tau}$$

and

$$(X + \delta Y)\sum_{h=0}^{\frac{c}{2}-1} \delta^{h\tau} \geq (\delta^2 X + \delta Y)\sum_{h=0}^{\frac{c}{2}-1} \delta^{h\tau}.$$

Now suppose that c is odd. Given a surplus-maximizing task-sharing arrangement, the player who is #2 takes the first task at the earliest opportunity whenever (10) holds. This condition is equivalent to

$$\begin{aligned} &2(X + \delta Y)\sum_{h=0}^{\frac{c-3}{2}} \delta^{h\tau} + X\delta^{(\frac{c-1}{2})\tau} \\ &\geq \delta(1+\delta)(X+Y)\sum_{h=0}^{\frac{c-3}{2}} \delta^{c\tau} + \delta(X+Y)\delta^{(\frac{c-1}{2})\tau} \\ &\iff \qquad 2(X + \delta Y)\frac{1 - \delta^{\frac{c-3}{2}\tau}}{1 - \delta^{\tau}} + X\delta^{(\frac{c-1}{2})\tau} \\ &\geq \delta(1+\delta)(X+Y)\frac{1 - \delta^{\frac{c-3}{2}\tau}}{1 - \delta^{\tau}} + \delta(X+Y)\delta^{(\frac{c-1}{2})\tau} \\ &\iff \qquad \frac{1 - \delta^{\frac{c-3}{2}\tau}}{1 - \delta^{\tau}}[2(X + Y\delta) - \delta(1+\delta)(X+Y)] \\ &\geq \delta^{(\frac{c-1}{2})\tau}[\delta Y - (2-\delta)X] \end{aligned}$$

which always holds if $X \geq \frac{\delta}{2-\delta} Y$ since the left hand side is positive. This expression can also be written as

$$\left[\frac{2(X+Y\delta) - \delta(1+\delta)(X+Y)}{(\delta Y - (2-\delta)X)}\right] > \frac{\delta^{(\frac{c-1}{2})\tau}(1-\delta^{\tau})}{1-\delta^{\frac{c-3}{2}\tau}}.$$

Observe that

$$\frac{\delta^{(\frac{c-1}{2})\tau}(1-\delta^{\tau})}{1-\delta^{\frac{c-3}{2}\tau}}$$

is decreasing in c and

$$\lim_{c\to\infty} \frac{\delta^{(\frac{c-1}{2})\tau}(1-\delta^{\tau})}{1-\delta^{\frac{c-3}{2}\tau}} = 0.$$

Therefore, for each parameter configuration there exists a c^* such that (10) holds for $c \geq c^*$. ■

Proof of Corollary 6. Condition (10) can be written

$$\left[\frac{2(X+Y\delta) - \delta(1+\delta)(X+Y)}{(\delta Y - (2-\delta)X)}\right] > \frac{\delta^{(\frac{c\mu-1}{2})\frac{\tau}{\mu}}(1-\delta^{\frac{\tau}{\mu}})}{1-\delta^{\frac{c\mu-3}{2}\frac{\tau}{\mu}}}. \tag{12}$$

Note that

$$\lim_{\mu\to\infty} \frac{c\mu-1}{2}\frac{\tau}{\mu} = \lim_{\mu\to\infty} \frac{c\mu-3}{2}\frac{\tau}{\mu} = \frac{c\tau}{2}.$$

Since $\lim_{\mu\to\infty} \delta^{\tau/\mu} = 1$, the right-hand side of (12) tends to zero as each task gets divided into more and more subtasks. There exists some subtask division such that (12) holds. ■

Proof of Proposition 7. Let $\kappa \leq c$ be the number of tasks remaining at a given period t. The specified strategies depend on the parameter configuration. We identify four cases.

Case 1. Suppose that $X \geq \frac{\delta}{2-\delta} Y$. Consider the following strategies. If a player is occupied while the other player is free, then the player who is free takes one of the remaining tasks. If both players are free, then #1 passes and #2 takes one of the remaining tasks. To show that these strategies specify a SPE, we examine each type of subgame in turn.

First suppose that both players are idle. When $\kappa \geq 2$, then by Lemma 4, it is a best reply for #2 to take a task. Given that #2 takes a task, and given that a player always prefers for the other to take a task rather than performing it himself, then #1 strictly prefers to pass. If $\kappa = 1$, then for the parameter range specified, it is an equilibrium for agents to play the pure strategy equilibrium on roles. By construction of this equilibrium, when both players are free, it is a best reply for #1 to pass and for the second to take the last task.

Next, suppose that in period t, player i is free and player j is busy for the next α periods. When $\kappa \geq 2$, Lemma 4 implies that player i prefers starting the task in $\alpha - 1$ periods to waiting until player j is free:

$$X\delta^{\tau+\alpha-1} + Y\delta^{\tau+\alpha} + X\delta^{2\tau+\alpha-1} + \cdots$$
$$\geq \frac{1}{2}[X\delta^{\tau+\alpha} + Y\delta^{\tau+\alpha+1} + X\delta^{2\tau+\alpha} + \cdots]$$
$$+ \frac{1}{2}[Y\delta^{\tau+\alpha} + X\delta^{\tau+\alpha+1} + Y\delta^{2\tau+\alpha} + \cdots].$$

Clearly, player i prefers taking the task now to starting it in $\alpha - 1$ periods; this implies immediately that player i prefers taking the task at her first opportunity rather than waiting for player j to become free. When $\kappa = 1$, player i prefers to take a task now rather than waiting just one period for player j to become free: $X\delta^{\tau} \geq \frac{1}{2}X\delta^{\tau+1} + \frac{1}{2}Y\delta^{\tau+1}$ since this condition is equivalent to $X \geq \frac{\delta}{2-\delta}Y$. It follows immediately that i prefers to take the last task in period t rather than waiting α periods for player j to become free.

Case 2. Suppose that $X < \frac{\delta}{2-\delta}Y$ and that (10) holds at $c = 3$. Consider the following strategies. Let $\kappa \geq 2$ in period t. As in Case 1, if one player is free while the other is busy, then the free player takes one of the remaining tasks. If both players are free, then #1 passes and #2 takes one of the remaining tasks. Now let $\kappa = 1$. Agents play the one-task mixed strategy equilibrium when both agents are free. If player i is occupied while player j is not, then player i takes the last task. To show that these strategies specify a SPE, we examine each type of subgame in turn.

First suppose that both players are idle. When $\kappa \geq 2$, the strategies are best replies at every subgame for the same reasons as in Case 1. By Lemma 4, it is a best reply for #2 to take a task, which implies that #1 strictly prefers to pass. If $\kappa = 1$, for the parameter range specified it is an equilibrium for agents to play the mixed strategy equilibrium.

Next, suppose that in period t, player i is free and player j is busy for the next α periods. When $\kappa \geq 2$, as in Case 1, Lemma 4 guarantees that player i prefers taking the task now to waiting for player j to become free. When $\kappa = 1$, by construction of the mixed strategy equilibrium, player i is indifferent between taking the last task in period t and waiting one period for player j to become free and for the mixed strategy equilibrium to be played,

$$X\delta^{\tau} = \delta Q(i),$$

where $Q(i)$ is the expected payoff of the one-task mixed strategy equilibrium. In particular, it is a best reply for player i to take the task. It follows immediately that player i would take the task if she had to wait α periods for player j to become free.

Case 3. Suppose that $X < \frac{\delta}{2-\delta} Y$. If c is odd, suppose that (10) does not hold at c. If c is even, suppose that (10) does not hold at $c-1$. Consider the following strategies. If c is even, then in period 0, #1 passes and #2 takes a task, and suppose without loss of generality that the player who takes a task in period 0 is player j. If c is odd, then in period 0, player i takes a task and player j passes, regardless of their positions in the permutation. If, in period t, κ is even and both players are free, then #1 passes and #2 takes a task. If κ is odd and both players are free, then player i takes a task and player j passes, regardless of their positions in the permutation. If, in period t, player i is free while player j is busy for the next α periods, then player i takes one of the remaining tasks at her first opportunity. If player j is free while player i is busy for the next α periods, then player j takes a task at her first opportunity if κ is even. If κ is odd, then player j passes whenever

$$\mathbf{I}[\kappa \geq 3] \cdot \sum_{m=1}^{\kappa-2} X\delta^{m\tau} + X\delta^{\frac{\kappa+1}{2}\tau} < \mathbf{I}[\kappa \geq 3] \cdot \sum_{m=1}^{\kappa-2} X\delta^{m\tau+\alpha+1} + Y\delta^{\frac{\kappa+1}{2}\tau+\alpha}. \quad (13)$$

Otherwise player j takes a task at her first opportunity. To show that these strategies specify a SPE, we examine each type of subgame in turn.

First, suppose that both players are free. If κ is even, then by Lemma 4, it is a best reply for #2 to take a task, which implies immediately that it is a best reply for #1 to pass. If κ is odd, player i can do no better to take a task now given that player j will pass; indeed, passing for player i would simply postpone benefits to a subsequent period:

$$X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + \cdots > X\delta^{\tau+1} + Y\delta^{\tau+2} + X\delta^{2\tau+1} + \cdots.$$

It is credible for player j to pass given that player i will concede. Since (10) does not hold at the start of the game (at c if c is odd, or at $c-1$ if c is even), then by Lemma 4, (10) also fails for any continuation with an odd number of tasks,

$$\begin{aligned} X\delta^{\tau} &+ Y\delta^{\tau+1} + X\delta^{2\tau} + \cdots \\ &< \frac{1}{2}[X\delta^{\tau+1} + Y\delta^{\tau+2} + X\delta^{2\tau+1} + \cdots] \\ &\quad + \frac{1}{2}[Y\delta^{\tau+1} + X\delta^{\tau+2} + Y\delta^{2\tau+1} + \cdots], \end{aligned}$$

and since $X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + \cdots > X\delta^{\tau+1} + Y\delta^{\tau+2} + X\delta^{2\tau+1} + \cdots$, it follows immediately that

$$X\delta^{\tau} + Y\delta^{\tau+1} + X\delta^{2\tau} + \cdots < [Y\delta^{\tau+1} + X\delta^{\tau+2} + Y\delta^{2\tau+1} + \cdots].$$

This means that it is a best reply for player j to pass even as second in the permutation, given that she expects player i to take a task first.

Second, suppose that i is free and that j is busy for the next α periods. If κ is even, as in Cases 1 and 2, Lemma 4 guarantees that player i prefers taking the task now to waiting for player j to become free. If κ is odd, then it is also a best reply for player i to take a task at her first opportunity. Indeed, were she to wait for player j to become free, player j would always pass and wait for player i to take the next task.

Finally, suppose that player j is free while player i is busy. If κ is even, then as before, Lemma 4 guarantees that player j prefers taking the task now to waiting for player i to become free. If κ is odd and $\kappa \geq 3$, then player j may or may not take a task. If player j were to wait, player i would take task immediately after becoming free. Therefore, player j waits for i if the benefits from i doing an extra task outweigh the cost of waiting,

$$X\delta^{\tau}+Y\delta^{\tau+\alpha}+X\delta^{2\tau}+Y\delta^{2\tau+\alpha}+\cdots+Y\delta^{\frac{\kappa-1}{2}\tau+\alpha}+X\delta^{\frac{\kappa+1}{2}\tau}$$
$$< Y\delta^{\tau+\alpha}+X\delta^{\tau+\alpha+1}+Y\delta^{2\tau+\alpha}+X\delta^{2\tau+\alpha+1}+\cdots+X\delta^{\frac{\kappa-1}{2}\tau+\alpha}+Y\delta^{\frac{\kappa+1}{2}\tau+\alpha}$$
$$\Longleftrightarrow X\delta^{\tau}+X\delta^{2\tau}+\cdots+X\delta^{\frac{\kappa+1}{2}\tau} < X\delta^{\tau+\alpha+1}+X\delta^{2\tau+\alpha+1}+\cdots+Y\delta^{\frac{\kappa+1}{2}\tau+\alpha},$$

which is exactly the condition given in (13).

Case 4. Suppose that $X < \frac{\delta}{2-\delta}Y$. If c is odd, suppose that (10) holds at c but not at 3. If c is even, suppose that (10) holds at $c-1$ but not at 3. By Lemma 4, there is some odd number of tasks c^* for which (10) holds at $c \geq c^*$ but not for $c < c^*$. Consider the following strategies. If, in period t, both players are free, κ is even, or κ is odd and $\kappa \geq c^*$, then #1 passes, and #2 takes a task. Suppose without loss of generality that player j is the player who performs the even-numbered tasks. If κ is odd and $\kappa < c^*$, then player i takes a task and player j passes, regardless of their positions in the permutation. If, in period t, player i is free while player j is busy for the next α periods, then player i takes one of the remaining tasks at her first opportunity. If player j is free while player i is busy for the next α periods, then player j takes a task at her first opportunity if κ is even or if κ is odd and $\kappa \geq c^*$. If κ is odd and $\kappa < c^*$, then player j passes whenever (13) holds.

For each of the subgames, it was shown in either Case 2 or Case 3 that the relevant strategies are best replies. ■

Proof of Lemma 5. By induction. We first show that the result holds for $c = 2$. There are two possible symmetric specifications in the $c-1 = 1$ task subgames, depending on the parameter configuration. With $X \geq \frac{\delta}{2-\delta}Y$, agents play the pure strategy equilibrium on roles, while with $X \leq \frac{\delta}{2-\delta}Y$, agents play a mixed strategy equilibrium. We must show that for each of these continuations the SPE in the two-chore game is the surplus-maximizing SSPE; i.e., we must show that at $t = 0$, #2 takes a task while the other player takes the remaining task at $t = 1$.

Consider the first specification in the one-task subgame where agents play the pure strategy equilibrium on roles. We start by establishing that once the first task is taken, the remaining player takes the second task without delay. Without loss of generality, suppose that player i has taken the first task in period 0. Player j prefers taking a task immediately to waiting for player i to become free and having a probability of $\frac{1}{2}$ that player i will take the last task as second in the permutation whenever

$$\begin{aligned} X\delta^{\tau} &\geq \delta^{\tau-1}\left(\frac{1}{2}X + \frac{1}{2}Y\right)\delta^{\tau} \\ &\iff X(2-\delta^{\tau-1}) \geq \delta^{\tau-1}Y \end{aligned}$$

which holds since, under the first specification,

$$X \geq \frac{\delta}{2-\delta}Y \geq \frac{\delta^{\tau-1}}{2-\delta^{\tau-1}}Y.$$

Since player j prefers to take the second task immediately to postponing the benefits of the task by waiting $1 \leq r \leq \tau - 1$ periods while player i is still busy, player j takes the second task in period 1 whenever player i takes the first task in period 0. Player i, in turn, takes the first task as second in the permutation by Lemma 4.

Consider the second specification in the one-task subgame, where players play the mixed strategy equilibrium when both are idle. Let $Q(i)$ be the ex ante payoff from playing the mixed strategy equilibrium in the one-task subgame. Assuming again without loss of generality that player i has taken the first task, it is a best reply for player j to take a task at her first opportunity rather than wait for player j to be free whenever

$$X\delta^{\tau} \geq \delta^{\tau-1}Q(i) = \delta^{\tau-1}\delta^{\tau-1}X.$$

As in the previous specification, player j also prefers taking the task at her first opportunity rather than waiting $1 \leq r \leq \tau - 1$ periods while player i is still busy. Also as in the previous specification, Lemma 4 guarantees that #2 takes the first task.

This establishes that in the $c = 2$ game, a symmetric continuation in the $\kappa = 1$ subgames where both players are available ensures that the only SPE of the two-chore game is the surplus-maximizing SSPE: the strategies are symmetric and the players share tasks without delay.

Now, assuming that the result is true in a game with $c - 2$ tasks, we show that it is true for the c-chore game. By assumption, the unique symmetric SPE of the $c - 2$ game is the surplus-maximizing SSPE and the continuation strategies in the $c - 1$ subgames are symmetric. Note that if in period 0 of the c-chore game #2 takes the first task and the other player follows by taking the next task at her first opportunity, then the strategies must specify task-sharing for the remainder of the game, and the SPE of the c-chore game is the surplus-maximizing SSPE. Indeed, since all $c-2$ subgame continuations feature task-sharing as the unique SPE, players can do no better than to continue sharing tasks without delay once the first two tasks have been assigned. Further, since Lemma 4 holds whenever c is even, it follows that #2 takes a task in period 0 as long as the other player takes the next task at her first opportunity. Therefore, to establish the result, we need only show that if player i (say) has taken the first task in period 0, then player j takes a task in period 1.

Player j's decision of whether to take the second task once player i is busy depends on the strategies specified were player j to wait for both players to be free. Given that the strategies in the $c-1$ subgame continuations are symmetric, the actions specified to assign the first task in the subgame can only depend on the player's position in the permutation, not on her identity. Given that these strategies are part of a SPE, it must be that #1 passes, while #2 either takes the next task or randomizes between passing and taking the task. To see that this is true, consider the other possible action combinations in turn. It cannot be that both players pass on the assignment of the task: this postpones the payoff from all tasks being done without any compensating benefit. It cannot be that #2 passes while #1 takes the task. If #1 is willing to take a task rather than wait for the next period where there is a $\frac{1}{2}$ probability that she will be second in the permutation, then it must be that

$$\delta^{\tau}X + V^2(c-2) \geq \frac{1}{2}\delta\,[\delta^{\tau}X + V^2(c-2)] + \frac{1}{2}\delta\,[\delta^{\tau}Y + V^1(c-2)],$$

where $V^1(c-2)$ ($V^2(c-2)$) is the sum of benefits from task-sharing on the last $c-2$ chores when the player under consideration takes on the first (second) of the last $c-2$ tasks. Similarly, if #2 is willing to pass, it must mean that

$$\delta^{\tau}X + V^2(c-2) < \frac{1}{2}\delta\,[\delta^{\tau}X + V^2(c-2)] + \frac{1}{2}\delta\,[\delta^{\tau}Y + V^1(c-2)]$$

which is a contradiction. It cannot be that #2 passes while #1 randomizes between taking the task and going on to the next period. Letting $Q(c-1)$ represent the ex ante benefit from playing these strategies for assigning the first task in the $c-1$ task subgame, #1 must be indifferent between taking a task now and going on to the next period, which requires

$$\delta^{\tau}X + V^2(c-2) = \delta Q\,(c-1)$$

while #2 being willing to pass requires

$$\delta^{\tau}X + V^2(c-2) < \delta Q\,(c-1)$$

which is a contradiction. This analysis implies that in symmetric SPE strategies of the $c-1$ subgame continuation, it is not possible for #2 to pass for sure. If #2 either takes the task or uses a mixed strategy, then from the analysis of the one-chore game, it is optimal for #1 to pass. Therefore, there are two cases to consider.

First consider the case where in the $c-1$ chore continuation where both players are free, the strategies specify that #1 passes and #2 takes a task. This means that in the $c-1$ subgame continuation, (10) holds: a player prefers taking the first task right away rather than waiting one period and having a probability of $\frac{1}{2}$ that the other player will take this task. It follows directly that, once i (say) is busy with the first task in the c-chore game, j prefers taking the second of c tasks (the first of the remaining $c-1$ tasks) in period 1 rather than waiting $\tau - 1$ periods and having a probability of $\frac{1}{2}$ that player i will take this task.

Next consider the case where in the $c-1$ chore continuation where both players are free, the strategies specify that #1 passes and #2 randomizes between passing and taking a task. By construction, in the $c-1$ continuation #2 is indifferent between taking the first task and waiting exactly one period and having a probability $0 < p < \frac{1}{2}$ that the other player will take this task. It follows directly that, once i (say) is busy with the first task in the c-chore game, j prefers taking the second of c tasks (the first of the remaining $c-1$ tasks) in period 1 rather than waiting $\tau - 1$ periods for i to take the next task with probability p.

Therefore, once #2 has taken the first task in period 0, the other player follows immediately by taking the second task in the following period. By Lemma 4, the player who is second in the permutation in period 0 will take the first task. Given that the unique SPE in the $c-2$ subgame continuations is the surplus-maximizing SSPE, it follows that the unique symmetric SPE in the c-chore game is the surplus-maximizing SSPE as well. ■

Proof of Proposition 10. Suppose that $n-1$ players are busy each with one chore and consider the decision of the nth player whether or not to undertake the last task. For player n to take this task, it is sufficient that doing so be better than waiting for the task to be done by the first player to become free. If the first $n-1$ tasks were taken in the first $n-1$ periods then player n takes the task when

$$\begin{aligned} X\delta^{\tau+(n-1)} &\geq Y\delta^{2\tau} \\ \iff &X \geq Y\delta^{\tau-(n-1)} \end{aligned} \tag{14}$$

which is true by assumption.

Next, consider the player who is last in the permutation when $n-2$ players are busy with a task. She undertakes a task when

$$X\delta^{\tau+(n-2)} + Y\delta^{\tau+(n-1)} \geq \max\left\{Y\delta^{\tau+(n-1)} + X\delta^{\tau+n}; Y\delta^{\tau+(n-1)} + Y\delta^{2\tau}\right\}.$$

Since it is always better to do a task earlier rather than later, this reduces to

$$X\delta^{\tau+(n-2)} \geq Y\delta^{2\tau} \qquad \Longleftrightarrow \qquad X \geq Y\delta^{\tau-(n-2)}$$

which is implied by the condition $X \geq Y\delta^{\tau-(n-1)}$.

In general, a player at position $h+1 = 2, \ldots, n-1$ in the queue takes a chore as last in the permutation whenever

$$X\delta^{\tau+h} + \sum_{\kappa=h+1}^{n-1} Y\delta^{\tau+\kappa} \geq Y \sum_{\kappa=h+1}^{n-1} \delta^{\tau+\kappa} + Y\delta^{2\tau}$$

which is equivalent to $X \geq Y\delta^{\tau-h}$ and which holds since $X \geq Y\delta^{\tau-(n-1)} \geq Y\delta^{\tau-h}$. Finally, for the player who would be the first to take a task, we have

$$X\delta^{\tau} + \sum_{\kappa=1}^{n-1} Y\delta^{\tau+\kappa} \geq Y \sum_{\kappa=1}^{n-1} \delta^{\tau+\kappa} + Y\,\delta^{2\tau+1}$$

which is equivalent to $X \geq Y\delta^{\tau+1}$, and which holds. ■

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