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Multiperson Bargaining over Two Alternatives

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We present a continuous-time model of multiperson bargaining with incomplete information, where only two agreements are possible. We show that under mild informational assumptions, players behave as if they were representing the other players who prefer the same alternative, individually playing a war-of-attrition against a conglomerate player of the opposite side. We also provide a characterization of the perfect Bayesian equilibria in undominated strategies of the game, which yields a unique outcome in most cases. *Journal of Economic Literature* Classification Numbers: C72, C78, D72, D74. © 1996 Academic Press, Inc.

1. INTRODUCTION

Consider a large group of individuals that have to agree on a choice from a set of possible alternatives. Faced with this challenge, civilized agents try to resolve the conflicts arising from their differing preferences by negotiation. Looking at the process of the resolution of conflict from the strategic point of view, the relevant questions in this context are: Will they reach an agreement? If yes, which will be the alternative finally chosen? How efficient is the bargaining procedure? How do the answers to the previous questions

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vary as the collection of agents (preferences) change? This paper takes a small step in the direction of answering these questions.

The usual approach in the literature on multilateral bargaining problems focuses attention on the study of strategic coalition formation, a problem that is undoubtedly as interesting as it is difficult. In negotiations with three or more players, either by simple use of a (possibly qualified) majority rule or via strategic threats, coalitions may have the power to impose an agreement on the rest of the parties. This is not the case in two-party negotiations where the acceptance of both negotiators (that is, consensus) is necessary for an agreement. Not surprisingly, this fundamental difference between two-party and multilateral bargaining problems is the basic obstacle in extending the Nash Program, of providing noncooperative foundations to cooperative solutions of bargaining games, to more than two players. The main efforts in the literature have concentrated on proposing specific complete information extensive forms or refinements of the Nash equilibrium concept that may provide some generalization of Rubinstein's (1982) results. These results do not extend even to the simplest three-player models of noncooperative bargaining with complete information. Herrero (1985) analyzes the three-player version of Rubinstein's game: players alternate in making offers on how to split a unit of surplus until all agree on some proposal. Under such a procedure any partition can be supported as the result of subgame perfect equilibrium strategies. It is true that there is a unique subgame perfect equilibrium in *stationary* strategies, however a model of bargaining where the players do not condition their actions on the past is not very intuitive. Binmore (1985) and Bennett and Houba (1992) propose two versions of the Nash bargaining solution with a noncooperative implementation for three-person bargaining problems in which only two-player coalitions can achieve gains. Such situations, labeled three-player three-cake problems, could arise if agreement can be reached by majority rule: any two-player coalition has the power to share the surplus generated by excluding the third player. The use of Rubinstein's results for bargaining within a (two-player) coalition is crucial in three-player three-cake problems, thus generalizations to more players are far from obvious. For an arbitrary number of players very specific extensive forms and refinements of the equilibrium concept are necessary in order to get sensible results. Some papers along this line are Asheim (1992), Chae and Yang (1988), Chatterjee *et al.* (1993), Rausser and Simon (1991), and Yang (1992).

Our work departs slightly from the standard approach. Instead of looking for a sufficiently constrained procedure to preserve uniqueness, we relax some of the procedural assumptions. At the cost of considering only two possible agreements, we obtain several benefits. First, we can characterize for *any number of players* an equilibrium that is *unique* for a relevant range of parameters, using a procedure where the *timing of offers is endogenous.*

Moreover, we can incorporate *incomplete information* about the players' preferences. These results shed little light on the intricacies of coalition formation and compromises, but they do improve our understanding of strategic behavior in negotiations with only two alternatives.

The assumption of two disciplined groups, each supporting a proposal, and the restriction of negotiation to these two proposals seems to be a good description of many multilateral negotiations. Appointing candidate A or candidate B to the Supreme Court, accepting or rejecting an amendment to the Constitution, ratifying or not ratifying a treaty, locating a public project in city A or in city B, etc., are all examples of public choice problems that, in democratic societies, need to be resolved by groups of individuals. Clear examples of our framework are legislative bodies made up of only two parties, each of which usually supports a distinct alternative. Moreover, even in legislatures with two large parties and several smaller ones, a game such as ours may arise since the smaller parties must decide on which of the large parties' proposals to support.

Imagine now that in an arbitrary negotiation, after a period of tentative talks, only two proposals remain on the table. Players will gather in two groups (coalitions, unions, political parties, etc.), each supporting one of the two alternatives. The members of each group agree that their proposal is better than their opponents' but they may disagree on how bad the latter is relative to their preferred alternative and relative to the possibility of an impasse. Each of the two groups meets *separately,* considering the proposal of their opponents. They privately argue whether they should withdraw their proposal or whether they should hold out, thereby incurring delay costs, in the hope that the other side will give in soon. The process continues until a sufficiently large number of members of one of the groups has declared a willingness to accept its less preferred alternative. At that point, one of the proposals has gathered sufficient support to become the outcome which is agreed upon.

The crucial assumption in the above scenario is that players from one group cannot observe the individual actions of players from the other group. Unfortunately, it is absolutely necessary for our results. Players need not know, however, the actions of their own comrades. Restricting attention to this class of scenarios, we are able to generalize our results on the war of attrition (Ponsati and Sákovics, 1995) to obtain a full characterization of the set of outcomes that can be supported by perfect Bayesian equilibria in undominated strategies, and thus we can give very precise answers to the questions posed at the outset of this paper. If the players' preferences are compatible, that is, there are not too many of them who prefer disagreement to their less preferred alternative, then they will always reach agreement. In some cases this agreement is reached immediately, while in the remaining cases the outcome is a continuous distribution of concession dates, where the most stubborn players concede arbitrarily late. We show that players behave as if they were representing their comrades, individually playing a war-of-attrition against a conglomerate player of the opposite side.

In continuation, we describe our model in detail in Section 2. Section 3 presents and discusses the results we obtained, while Section 4 contains some final remarks. The proofs of our intermediate results are provided in the Appendix.

2. The Model

Consider a game with N players negotiating over two feasible alternatives, A and B. The players are classified into two nonempty groups, **A** (containing k members) and **B** (containing $N - k$ members), depending on whether they prefer agreement at A or B, respectively. The composition of these groups is assumed to be common knowledge. Let us denote a generic agreement by X (similarly, denote a generic group by $\mathbf{X}$). Then Player i's preferences can be described as follows[1]: if she is of type s—a privately known parameter (normalized reservation value)—and she prefers outcome X, then if decision X is reached at time t, her utility is $(1 - s)e^{-t}$, whereas if the opposite decision is reached, her payoff is $(-s)e^{-t}$. Perpetual disagreement gives 0 to all the players.

The players entertain beliefs about each other's type. We assume that these a priori beliefs are correct and common knowledge among the players. Moreover, they can be represented by a (joint) probability distribution $F(\mathbf{s})$, with positive smooth partials $F_i(\mathbf{s})$ over the set $\times_{i=1}^{k}[i_L, i_H]$ for the agents preferring A, and by $G(\mathbf{z})$, with positive smooth partials $G_j(\mathbf{z})$ over the set $\times_{j=1}^{N-k}[j_L, j_H]$ for the agents preferring B. That is, each player's type distribution has an interval support. For simplicity, we require that players who prefer the same alternative have common support; however, this assumption can be relaxed. We impose two assumptions about these intervals: (i) $i_L, j_L < 0$; (ii) $i_H, j_H \leq 1$. That is, with positive probability every player is of a type which derives a positive utility even if his less preferred alternative prevails, and there are no players who derive negative utility even if their preferred alternative is agreed upon. The only substantive restriction we impose on the distributions is that the random vectors described by $F(.)$ and $G(.)$ are independent.

The game is played in continuous time, starting at $t = 0$. The players

[1] We use these preferences for simplicity. Actually, we only need the following restrictions on preferences (assume that we deal with a player preferring A): $u(\text{B}, s) \geq u(\text{Disagreement})$ iff s is nonpositive; $u(\text{A}, s) > u(\text{B}, s)$ for all s; $\partial u(X, s)/\partial s < 0$; and $u(\text{B}, s') - u(\text{B}, s) \geq u(\text{A}, s') - u(\text{A}, s)$ for $s' \geq s$.

have a unique action available to them, although they can freely choose its timing: they can *yield.* A player from group **X** starts out with a vote in favor of alternative X. This situation persists until she yields, from which point on her vote counts for both agreements. Let us denote by Q ($>N/2$) the number of votes needed for an alternative to be chosen. Then an agreement is reached when one of the alternatives has at least Q votes.[2] (In the case that both alternatives get the necessary number of votes at the same time, a lottery is used to decide on the outcome. The only requirement we impose is that the lottery cannot assign probability one to either alternative.) We make an important informational assumption: only the members of the same group may observe when and whether a player has yielded. Players in the opposite group can only tell whether there have been sufficient concessions in this group for their alternative to be chosen.

A partial history for each player, h_t, is the set of previous yielding dates of the players in her group. Thus, a strategy for player i, σ_i, is a (measurable) function from his type and the partial history to the (possibly infinite) time of his concession. We are more than aware of the importance of selecting the appropriate set of allowable strategies in continuous time games.[3] However, in order not to divert the reader's attention, and since we give a characterization in terms of equilibrium *outcomes,* we skip the proof that there exists such a set and that all the strategies used in the proofs belong to it. Instead, we simply claim that after any partial history a strategy profile, σ, and a type profile, $\mathbf{q}$, uniquely determine on outcome with agreement on $X(\sigma(\mathbf{q}, h_t))$ at time $t(\sigma(q, h_t))$.

A belief system, $\beta_i(h_t)$, for player i in a candidate equilibrium is the (joint) probability distribution over the types of the rest of the players. Given a strategy-belief profile (σ, β), let $V_i^s(\sigma, \beta_i, h_t)$ denote the expected payoff to player i of type s conditional on his beliefs after h_t, that is,

$$V_i^s(\sigma, \beta_i, h_t) = \int U_i(X(\sigma(s_i, \mathbf{r}, h_t)), s, t(\sigma(s_i, \mathbf{r}, h_t)))\, d\beta_i(h_t)(\mathbf{r}),$$

where $\mathbf{r}$ denotes an $(N-1)$-tuple of types and U_i denotes Player i's utility function, as described above.

A pair (σ, β) is a *Bayesian equilibrium* (BE) if and only if β is consistent with F and G according to Bayes' rule, and for all s and $i = 1, 2, \ldots, N$, $V_i^s(\sigma, \beta_i, h_0) \geq V_i^s((\sigma_i', \sigma_{-i}), \beta_i, h_0)$ for all σ_i'.

A pair (σ, β) is a *perfect Bayesian equilibrium* (PBE) if and only if for all t, β is consistent with σ, F, and G according to Bayes' rule, and for all s and $i = 1, 2, \ldots, N$, $V_i^s(\sigma, \beta_i, h_t) \geq V_i^s((\sigma_i', \sigma_{-i}), \beta_i, h_t)$ and also for all σ_i'.

[2] Equivalently, we could let Q be the fraction of cast votes needed for agreement and suppose that players abstain when they are willing to accept their less preferred alternative.

[3] For an excellent and exhaustive treatment of these difficulties, consult Stinchcombe (1992).

Finally, (σ, β) is a *perfect Bayesian equilibrium in undominated strategies* (PBEUS) if and only if it is a PBE and if for no s_i does there exist a σ_i' such that $U_i(X(\sigma'(s_i, \mathbf{r}, h_t)), s, t(\sigma'(s_i, \mathbf{r}, h_t))) \geq U_i(X((\sigma_i, \sigma'_{-i})(s_i, \mathbf{r}, h_t)), s, t((\sigma_i, \sigma'_{-i})(s_i, \mathbf{r}, h_t)))$ for all σ'_{-i} and $\mathbf{r}$, with at least one strict inequality.

3. The Solution

To fix ideas, we start with the two-person case. Note that in the present context an equilibrium outcome is a probability distribution over $\{A, B\} \times R_{+,0} \cup D$.

Proposition 1A (War-of-attrition, Ponsati and Sákovics, 1995). *If $Q = N = 2$, then the set of BE outcomes is characterized by the following*:

(i) *If $i_H \leq 0$ and $0 < j_H$ then, almost surely, i (and only i) concedes at 0 (and vice versa).*

(ii) *If $0 < i_H$ and $0 < j_H$ then strategies are such that $\sigma_i(s) = t$ if and only if $\varepsilon(t) = s$, and $\sigma_j(z) = t$ if and only if $\zeta(t) = z$, where ε: $(0, \infty) \to (i_L, 0]$ and ζ: $(0, \infty) \to (j_L, 0]$ are increasing, differentiable functions that uniquely solve the following system of differential equations*:

$$\begin{aligned} -\zeta'(t)g(\zeta(t)) &= (1 - G(\zeta(t)))\varepsilon(t) \\ -\varepsilon'(t)f(\varepsilon(t)) &= (1 - F(\varepsilon(t)))\zeta(t), \end{aligned} \tag{1A}$$

with boundary conditions $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$, $\lim_{t\to\infty} \varepsilon(t) = 0$, *and* $\lim_{t\to\infty} \zeta(t) = 0$.

(iii) *If $i_H \leq 0$ and $j_H \leq 0$ then the players use strategies characterized by the solutions to* (1A), *such that* $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$. *These include two equilibria where, almost surely, exactly one player concedes at zero. In the other equilibria, if $i_H < 0$ and $j_H < 0$ then* $\lim_{t\to T} (\varepsilon(t), \zeta(t)) = (i_H, j_H)$ *for some $T \leq \infty$; and if $i_H = 0$ or $j_H = 0$, then* $\lim_{t\to\infty} (\varepsilon(t), \zeta(t)) = (\eta, \phi)$, *for $i_L \leq \eta \leq i_H$, $j_L \leq \phi \leq j_H$, and $(\eta - i_H)(\phi - j_H) = 0$.*

Proof. See the end of this section.

Note that positive types prefer disagreement to any other outcome that gives them their less preferred alternative and therefore they would never yield in equilibrium. Thus, if one of the players is believed to have a positive type with positive probability, it is credible for him to claim that he will never yield. So, if his opponent is known to be willing to yield (case i), he can take advantage of her. If both can be unwilling to yield, we get the classical war-of-attrition behavior: the players try to screen each other's type by prolonging the game and thereby imposing a delay cost on their opponent (as well as themselves). Note that all players who derive positive utility even upon concession eventually yield in equilibrium.

If both players are known to prefer agreement on their less preferred alternative to disagreement we get again the wearing down strategies solving (1A) but with nonunique boundary conditions. In addition, there are also two additional equilibria. These new equilibria arise because no longer is it credible that a player never will concede. Therefore, if either of them believes after any history (i.e., time without concession) that the other will concede soon, it is optimal for her to hold on. But then her opponent is best off yielding as soon as possible. In our view, these two equilibria (especially given the existence of the other ones) are not very reasonable. However, in this paper we emphasize other issues, so we do not make a detour to "refine away" these unreasonable equilibria.

Next, we turn to the multiperson case, but for the time being we keep the requirement of unanimity for an agreement. Unless otherwise indicated, objects in bold letters indicate vectors that take the same value at each component.

PROPOSITION 1B (Unanimity). *If $Q = N$, the set of PBE outcomes is characterized by the following*:

(i) *If $i_H \leq 0$ and $0 < j_H$ then, almost surely, group $\boldsymbol{X}$ (and only group $\boldsymbol{X}$) gives in at zero (and vice versa).*

(ii) *If both $0 < i_H$ and $0 < j_H$ then strategies are such that $\sigma_i(s, h_t) = \max(t, t')$ if and only if $\varepsilon(t') = s$, and $\sigma_j(z, h_t) = \max(t, t')$ if and only if $\zeta(t') = z$, where ε: $(0, \infty) \to (i_L, 0]$ and ζ: $(0, \infty) \to (j_L, 0]$ are increasing, differentiable functions that uniquely solve the following system of (partial) differential equations*[4]:

$$-\zeta'(t) \sum_{j=1}^{N-k} G_j(\zeta(t)) = (1 - G(\zeta(t)))\varepsilon(t)$$
$$-\varepsilon'(t) \sum_{i=1}^{k} F_i(\varepsilon(t)) = (1 - F(\varepsilon(t)))\zeta(t), \tag{1B}$$

with boundary conditions $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$, $\lim_{t\to\infty} \varepsilon(t) = 0$, *and* $\lim_{t\to\infty} \zeta(t) = 0$.

[4] $G_j(\cdot)$ ($F_i(\cdot)$) denotes the derivative with respect to the jth (ith) coordinate of $G(\cdot)$ ($F(\cdot)$).

(iii) *If* $i_H \leq 0$ *and* $j_H \leq 0$ *then the players use strategies characterized by the solutions to* (1B), *such that* $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$. *These include two equilibria where*, *almost surely*, *exactly one group concedes at zero. In the other equilibria, if* $i_H < 0$ *and* $j_H < 0$ *then* $\lim_{t\to T} (\varepsilon(t), \zeta(t)) = (i_H, j_H)$ *for some* $T \leq \infty$; *and if* $i_H = 0$ *or* $j_H = 0$, *then* $\lim_{t\to\infty} (\varepsilon(t), \zeta(t)) = (\eta, \phi)$, *for* $i_L \leq \eta \leq i_H, j_L \leq \phi \leq j_H$ *and* $(\eta - i_H)(\phi - j_H) = 0$.

Remarkably, Proposition 1B is almost identical to Proposition 1A. The underlying idea is simple: since consensus is required for any resolution, all the players are pivotal; without the consent of any one of them there can be no agreement. Consequently, they play as if only their decision were relevant, conceding at the time they would concede if they could dictate all the votes of their group.[5] The only change between (1A) and (1B) is that the new LHS's are the derivatives (with respect to t) of multivariate distributions, and thus they are slightly more complex to write. We need to evaluate $G(.)$ and $F(.)$ at the same value for all coordinates ($G(\zeta(t))$ and $F(\varepsilon(t))$) due to the unanimity rule: since any player needs *all* the players from the other group to yield to obtain her preferred alternative, *all* the players from the opposite group need to have types below the critical value.

Note, however, that we had to impose perfection to retain the structure of the two-player result. This issue did not even come up with the previous proposition since, in the two-player game, upon any action the game immediately ends and consequently there are no subgames to worry about. (Given the continuity of the equilibrium strategies there are no zero probability events either, which explains the "small number" of the BE.) Perfection is necessary because when there are several players in a group, a (nonperfect) equilibrium could involve all of them conceding later than optimal, according to the philosophy that it does not matter what any one player does at t if some of the others do not concede by t (compare with a row of double-parked cars). In the Appendix (in the Proof of Lemma 1) we demonstrate that in a perfect equilibrium such unreasonable strategies do not arise.

Now, we are ready to state our result for the general case, where only a qualified majority is necessary for an agreement. Let S_{A} denote the set of strict subsets of **A** with at least $Q + k - N$ elements, and similarly by S_{B} denote the set of strict subsets of **B** with at least $Q - k$ elements.[6] Also, let S be an arbitrary set of players. Finally, we define $\mathbf{x}(S) = (x_1, x_2, \ldots,$

[5] Compare this with the behavior of a bidder in a second-price sealed-bid auction.

[6] Note that $Q + k - N$ $(Q - k)$ is the minimal number of votes from group **A** (**B**) needed for group **B**'s (**A**'s) proposal to prevail.

x_k) and $\mathbf{y}(S) = (y_1, y_2, \ldots, y_{N-k})$, where

$$x_i = \begin{cases} \varepsilon(t), \text{ if } i \in S, \\ i_H, \quad \text{otherwise,} \end{cases} \qquad \text{and} \qquad y_j = \begin{cases} \zeta(t), \text{ if } j \in S, \\ j_H, \quad \text{otherwise.} \end{cases}$$

PROPOSITION 1C (Qualified Majority). *If $Q < N$, the set of PBEUS outcomes is characterized by the following*:

(i) *If $i_H < 0$ and $0 < j_H$ then, almost surely, group* ***X*** (*and only group* ***X***) *gives in at zero* (*and vice versa*).

(ii) *If both $0 < i_H$ and $0 < j_H$, then strategies are such that $\sigma_i(s, h_t) = \max(t, t')$ if and only if $\varepsilon(t') = s$, and $\sigma_j(z, h_t) = \max(t, t')$ if and only if $\zeta(t') = z$, where ε: $(0, \infty) \to (i_L, 0]$ and ζ: $(0, \infty) \to (j_L, 0]$ are increasing, differentiable functions that uniquely solve the system of* (*partial*) *differential equations:*

$$\begin{aligned} &\zeta'(t)\left[(\#S_B - 1)\sum_{n=1}^{N-k} G_n(\zeta(t)) - \sum_{S\in S_B}\sum_{j\in S} G_j(\mathbf{y}(S))\right] \\ &\quad = \varepsilon(t)\left[1 + (\#S_B - 1)G(\zeta(t)) - \sum_{S\in S_B} G(\mathbf{y}(S))\right] \\ &\varepsilon'(t)\left[(\#S_A - 1)\sum_{n=1}^{k} F_n(\varepsilon(t)) - \sum_{S\in S_A}\sum_{j\in S} F_j(\boldsymbol{x}(S))\right] \\ &\quad = \zeta(t)\left[1 + (\#S_A - 1)F(\varepsilon(t)) - \sum_{S\in S_A} F(\boldsymbol{x}(S))\right], \end{aligned} \tag{1C}$$

where[7] *$\#S_A = \sum_{i=0}^{N-Q-1} C_{k,Q+k-N+i}$ and $\#S_B = \sum_{i=0}^{N-Q-1} C_{N-k,Q-k+i}$ are the number of elements of S_A and S_B, with boundary conditions* $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$, $\lim_{t\to\infty} \varepsilon(t) = 0$ *and* $\lim_{t\to\infty} \zeta(t) = 0$.

(iii) *If $i_H \le 0$ and $j_H \le 0$ then the players use strategies characterized by the solutions to* (1C), *such that* $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L = 0$. *These include three equilibria where, almost surely, at least one group concedes at zero. In the other equilibria, if $i_H < 0$ and $j_H < 0$ then* $\lim_{t\to T} (\varepsilon(t), \zeta(t)) = (i_H, j_H)$ *for some $T \le \infty$; and if $i_H = 0$ or $j_H = 0$, then* $\lim_{t\to\infty} (\varepsilon(t), \zeta(t)) = (\eta, \phi)$, *for $i_L \le \eta \le i_H$, $j_L \le \phi \le j_H$, and $(\eta - i_H)(\phi - j_H) = 0$.*

[7] Recall that $C_{n,k}$ is the number of ways one can choose k elements out of a set of n elements.

If one relaxes the unanimity requirement to a qualified majority rule, the formulas become even more complex but the results remain essentially the same. The key observation here is that even though players are not necessarily pivotal, there always is a positive probability that they are, indeed, so. Consequently, since they are indifferent if their concession is outcome-irrelevant, they strictly prefer to behave as if they were pivotal. The LHS still represents the probability density at t of the opposing team's capitulation. The only difference is that this event can come about in many ways now, because of the qualified majority rule. Consequently, the probability of a group concession can be calculated only in a more complex way (cf. the proof of Lemma 10 in the Appendix).

In order to keep out unreasonable equilibria where—when their concession is outcome irrelevant—players who prefer disagreement to concession still concede,[8] we had to further strengthen our equilibrium concept by ruling out dominated equilibrium strategies.

Before presenting the proofs, we introduce some notation and intermediate results. Lemmas 0–6 and 10 are proved in the Appendix. The rest are proved in Ponsati and Sákovics (1995). Let $H_X^{\sigma}(t)$ denote the probability that group **X** gives in before t according to σ. We will write $H_X(t)$ if it does not lead to confusion. Moreover, let $\Sigma_X(t) = H_X(t) - \lim_{\delta \to 0} H_X(t - \delta)$, that is, $\Sigma_X(t)$ is the probability that **X** gives in at t. In the following lemmas, (P)BE(US) means that we look at BE if $(Q=)N = 2$, PBE if $Q = N > 2$, and PBEUS if $Q < N$ (>2).

LEMMA 0 (Positive Types Out). *In no (P)BE(US) do types $s > 0$ ever concede. Moreover, no strategy of a type $s < 0$ prescribing concession at a finite time is dominated.*

Based on this lemma, from now on, we restrict our attention to negative types and refrain from verifying whether the proposed strategies are undominated.

LEMMA 1 (Simultaneous Gaps). *Every (P)BE(US) outcome can be supported by strategies such that if in any subgame the conditional probability that group* **X** *concedes in the interval $(t, t + \delta]$ is nil, then, almost surely,* **Y** *does not concede in the interval $(t, t + \delta]$ either, that is, $H_Y(t) = H_Y(t + \delta)$.*

This lemma is instrumental in later proofs. It is also very intuitive: the only reason for a group to hold out is to believe that the opponents will give in "soon." If it is known that they will not, it is better to concede as early as possible. This observation leads to the conclusion that, in equilib-

[8] It is not too hard to see that these are the only strategies that get eliminated by the additional refinement.

rium, there is no mass point of group concession probabilities after time zero.

LEMMA 2 (Continuity). *Every* (*P*)*BE*(*US*) *satisfying Lemma* 1 *yields continuous group concession probabilities. That is*, $\Sigma_X(t) = 0$ *for all* $t > 0$.

Let $T_X = \min\{t$, such that $H_X(t) = H_X(\infty)\}$. That is, T_X is the earliest time by which group **X** gives in with probability one, conditional on giving in ever.

LEMMA 3 (Symmetric Spread). *In every* (*P*)*BE*(*US*) *satisfying Lemma* 1,

(i) *if both* T_A *and* T_B *are positive then they are equal;*

(ii) *if* $T_X < \infty$ *and* $H_X(T_X) = 1$ *then* $T_X = 0$.

LEMMA 4 (Strict Monotonicity). *In every* (*P*)*BE*(*US*) *satisfying Lemma* 1, *unless the game ends at zero*, *the probability that either group concedes is positive in any time interval up to* T_X. *That is*, $H_X(t - \delta) < H_X(t)$ *for* $0 < \delta < t \leq T_X$.

LEMMA 5 (Independence of Comrades). *In every* (*P*)*BE*(*US*) *satisfying Lemma* 1, *strategies are independent of the actions of the rest of the players. Moreover*, *these strategies correspond to the ones the players would play if they were the dictator of their team. That is*, $\sigma_i(s, h_t) \equiv \sigma(s, t)$.

Lemma 5 is the key observation of this paper. It is this simple structure of the strategies that makes it possible to generalize the two-person results. To understand why this is true, note that, given the opponents' strategies, every player has a preferred time for his team's concession, which only depends on his type. During the game he will not learn more about the opponents, since he does not observe their individual concessions and since the opponents' types are independent of his comrades' types (about which he can learn). Assume that he sees a comrade concede (note that this is the only observable action). This will not change his preferred date for his teams' final concession since it does not have an effect on the opponents' behavior (because they do not observe it). Thus, it is optimal for him to concede at his preferred date, since conceding at any other time is either outcome irrelevant or suboptimal.

LEMMA 6 (Immediate Concessions). (i) *If* $Q = N$ *then in no BE satisfying Lemma* 1 *do both groups concede at* 0 *with positive probability*. *That is*, $H_A(0) \cdot H_B(0) = 0$.

(ii) *If* $Q < N$ *then*, *in every BE satisfying Lemma* 1, $H_X(0) \cdot H_Y(0) \in \{0, 1\}$.

The following lemma sets up the appropriate objective function that players maximize:

LEMMA 7 (Best Response Mapping). *An outcome can be supported by a* $(P)BE(US)$ *if and only if there exists a strategy profile* σ, *independent of* h_t, *such that for all* $i \in \boldsymbol{A} \cup \boldsymbol{B}$ *and all* $h_{t'}$ (*for BE* $t' = 0$ *only*),

$$\sigma_i(s, t') \in \arg\max_{t > t'} \int_{[t',t]} (1 - s)e^{-\tau}\, dH_Y(\tau) - (1 - H_Y(t))se^{-t}.$$

LEMMA 8 (Strategies Monotone in Type). *Every* $(P)BE(US)$ *satisfying Lemma* 1 *is such that for all* i, $s < s'$ *implies that* $\sigma_i(s) \le \sigma_i(s')$.

That is, agreeing with intuition, a player for whom the relative difference between the alternatives is small is willing to incur only a small delay cost in order to force a favorable outcome.

LEMMA 9 (Differentiability). *Every* $(P)BE(US)$ *satisfying Lemma* 1 *yields differentiable group concession probabilities. That is, there is a continuous function* $h_X(.)$, *such that for all* $t \in (0, T_X)$,

$$H_X(t) = H_X(0) + \int_{(0,t)} h_X(\tau)\, d\tau.$$

LEMMA 10 (Differential Equation System). *Every* $(P)BE(US)$ *satisfying Lemma* 1 *is such that for* t *in* $[0, T_X)$, $\varepsilon(.)$ *and* $\zeta(.)$ *are a solution to* (1C).

LEMMA 11 (Admission of Defeat). *In every* $(P)BE(US)$ *satisfying Lemma* 1,

(i) *if* $i_H < 0$ *and* $j_H < 0$ *then if the outcome is not immediate agreement then eventually both groups will concede.*

(ii) *if there is a positive probability that group* $\mathbf{Y}$ *will never give in* ($H_Y(\infty) < 1$), *then* $\sigma_i(s) < \infty$ *for all* $s_i < 0$ *in group* $\boldsymbol{X}$.

For brevity, we do not state the last lemma explicitly. It is stated and proved in Ponsati and Sákovics (1995).

LEMMA 12 (Existence and Uniqueness of Solutions to (1C) Satisfying the Respective Boundary Conditions). *Given as Lemma* 11 *in Ponsati and Sákovics (1995).*

We are now ready to complete the proofs of the propositions.

Proof of Proposition 1C. (i) By Lemmas 0 and 11 there is some $T < \infty$ such that all $i \in \mathbf{X}$ (and therefore the whole group) concede no later than T with probability 1. By Lemma 3(ii) the concession time must be $T = 0$. On the other hand, since by Lemma 0 there is a positive probability that group $\mathbf{Y}$ does not give in at zero, by Lemma 6(ii) it almost surely will not do so.

(ii) Since along the equilibrium path ε and ϕ, as well as ζ and ν, are equivalent, Lemmas 0, 1, and 10 imply that any equilibrium outcome must be characterized by a solution to (1C). Define $\Phi(\varepsilon(t)) = \zeta(t)$. That is, given a type in group **A**, Φ selects the type of group **B** that concedes at the same time along the equilibrium path. By Lemmas 3(i), 4, and 5, Φ is well defined and by Lemma 8 it is strictly increasing. Also, by the differentiability of ε and ζ, Φ is differentiable with respect to ε and thus $\Phi'(\varepsilon(t)) = \zeta'(t)/\varepsilon'(t)$. Substituting Φ into (1C) and eliminating $\varepsilon'(t)$, we get a single first-order differential equation for Φ in ε. By Lemmas 0, 3, 8, and 11, $\Phi(0) = 0$. With this boundary condition Φ is uniquely characterized. Now, since Φ is continuous and monotone, either $j_L \leq \Phi(i_L)$ or $i_L \leq \Phi^{-1}(j_L)$, and therefore $\varepsilon(0) = i_L$ and $\zeta(0) = \Phi(i_L)$ or $\varepsilon(0) = \Phi^{-1}(j_L)$ and $\zeta(0) = j_L$, respectively. With these initial conditions (1C) has a unique solution.

It is immediate to check that, given his Bayesian beliefs, player i of type $\varepsilon(t)$ (j, of type $\zeta(t)$) cannot increase his payoff by conceding at $t' \neq t$. Thus the proposed strategy is indeed a PBEUS.

(iii) If all the players of a group concede at zero, this is a best response independent of the opponents' strategies, since none of the players are pivotal. This supports the three immediate agreements. If there is a positive probability of later agreement, then by Lemma 10 the strategies have to satisfy (1C). We only need to verify the terminal conditions. Lemma 6(ii) implies that $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$. Lemma 11(i) yields the terminal conditions when both i_H and j_H are less than zero. When (at least) one of the upper ends of the supports is zero, by Lemma 11(ii) it must be that $(\lim_{t\to\infty} \varepsilon(t) - i_H)(\lim_{t\to\infty} \zeta(t) - j_H) = 0$. Lemma 12 characterizes the solutions to (1C) satisfying these conditions. ■

Proof of Propositions 1A *and* 1B. For parts (i) and (ii) the previous proof applies. In part (iii) it is straightforward to verify that the immediate concessions can be supported as (P)BE with appropriate beliefs from at least one player of the opponents. Since Lemma 6(i) also applies, only one of the groups can give in at zero. If there is some player conceding after zero in both teams, then by Lemma 10 those strategies have to satisfy (1B). ■

4. Conclusion

In this paper we have departed from the standard approach to multiperson bargaining problems. By restricting attention to situations where only two agreements are possible, and by imposing an important informational restriction, we have been able to extend a robust, two-person result to the case of many players in a fairly straightforward manner. The key issue here—apart from existence, and the fact that we have a full characterization theorem—is the essential uniqueness of equilibrium outcomes in the framework of an n-person incomplete information bargaining game. It is true that we obtain uniqueness only when the players of at least one group attach some positive probability to the event that the opposing group will never yield, but we consider that this is the most relevant case to examine, since people are likely to entertain doubts (about the reliability of their information, the rationality of their opponents, etc.) that lead them to such beliefs.

In order to see *why* we get uniqueness (at least, when the players' beliefs are sufficiently pessimistic), let us outline first what the usual reasons that lead to multiplicity are.

In the majority of the n-person bargaining problems which have been analyzed, the driving force behind nonuniqueness is that deviations from a potential equilibrium strategy are weakened by the fact that a player can be rewarded by the rest of the players for rejecting the deviant offer. In some sense, the current proposer is always faced with an implicit coalition of the others. In our case this phenomenon does not arise because we have only two possible agreements and thus there is no room for compensation (side payments are not allowed). Two-sided incomplete information can give rise to multiplicity in two ways: either in equilibrium one of the players eventually reveals her type and thus we get into a (sub-)game of one-sided incomplete information that in general is known to have multiple equilibria whenever both players are active; or there exist subforms that are not reached in equilbrium and the players' posteriors in those subforms cannot be restricted sufficiently to restore uniqueness. In the war-of-attrition and its multiperson variants, since the distribution of types is continuous, equilibrium strategies are continuous in type and so types do not reveal themselves[9] (they only do so by conceding and thus leaving the game). Neither do these games have zero probability events (at every date some type is supposed to concede) so that Bayes' rule always can be applied. However, since our formulation allows for simultaneous concessions, there are other potential reasons to have an increased number of equilibria. Since noncooperative solution concepts do not require collective rationality, strategy profiles that involve players using collectively suboptimal strategies in the belief that their move is inconsequential can be sustained as Nash equilibria, disregarding the externality that their move contributes to rendering the other players' "bad" move optimal. Fortunately, using some well-established refinements, we can correct for this deficiency of the equilibrium concept.

A word on the information structure of the model. The assumption that individuals do not publicly announce their concession until it yields an

[9] Nor others, given our informational assumptions.

agreement is useful, because it simplifies the information updating that each player must perform as time goes by without an agreement. If individuals publicly (that is, in front of players from the opposing group) announced their willingness to concede, players would have to update their beliefs in a discontinuous way. This in turn would cause discontinuous changes in the strategies that would affect the beliefs again, and so on. On the other hand, since no agreement is reached until one of the alternatives gets sufficient support, it does not seem to be unreasonable to assume that players do not publicly announce their willingness to concede. In real life negotiations, unless a concession is decisive, the readiness of an individual to concede may circulate as a rumor, but usually it is not couched in a credible public announcement.

APPENDIX

Here, we proceed to prove some of the intermediate lemmas presented in the text. Lemmas 7–9, 11, and 12 are practically identical to Lemmas 6–8, 10, and 11 of Ponsati and Sákovics (1995), so we do not repeat the proofs.

LEMMA 0 (Positive Types Out). *In no $(P)BE(US)$ do types $s > 0$ ever concede. Moreover, no strategy of a type $s < 0$ which prescribes concession at a finite time is dominated.*

Proof. Under the unanimity rule, every player is pivotal. Since a player with positive type obtains negative utility upon any agreement that gives him his less preferred alternative, and the disagreement outcome gives him zero, he strictly prefers not to concede. Under majority rule, concession is weakly dominated. The proof of the second statement we leave to the reader. ■

LEMMA 1 (Simultaneous Gaps). *Every $(P)BE(US)$ outcome can be supported by strategies such that if in any subgame the conditional probability that group $\boldsymbol{X}$ concedes in the interval $(t, t + \delta]$ is nil, then, almost surely, $\boldsymbol{Y}$ does not concede in the interval $(t, t + \delta]$ either, that is, $H_Y(t) = H_Y(t + \delta)$.*

Proof. If a player knows that her opponents do not give in during the interval $(t, t + \delta]$, then making a concession at any t' in $(t, t + \delta]$ that is decisive with positive probability cannot maximize her payoff: since payoffs are discounted, any player who obtains a positive payoff from concession will increase her payoff if she does so a bit earlier. Assume that she makes the last nondecisive (because other players are conceding at the same time)[10] concession at some t' in $(t, t + \delta]$. Then, in the subgame where at

[10] If the concession is non-decisive because there are at least $N - Q + 1$ players of the group who concede after $t + \delta$ with probability one, then conceding at t would be an equivalent best response.

$t' - \rho > t$ she observes that she is the pivotal member of her group, conceding at $t' - \rho/2$ strictly dominates conceding at t'. Thus any PBE strategy has to prescribe an earlier concession than t'. We proceed by induction. Assume that if, at $t' - \rho/2$, $n - 1$ players who planned to concede at t' observe that they are jointly pivotal, then one of them could profitably deviate, bringing forward the final agreement. Take an equilibrium that has n players conceding at t'. If one of them deviates and concedes at $t' - \rho$ then this will bring forward the final agreement, benefiting the deviator. Therefore no player can concede at t' in $(t, t + \delta]$. ■

LEMMA 2 (Continuity). *Every (P)BE(US) satisfying Lemma* 1 *yields continuous group concession probabilities. That is,* $\Sigma_X(t) = 0$ *for all* $t > 0$.

Proof. Assume that $H_X(.)$ has a jump at some $t > 0$. Assume, moreover, that a player of group **Y** intends to concede before t and there is a positive probability that his concession is decisive. The gain in the undiscounted expected payoff that he obtains if he waits and concedes after t is strictly greater than 0. The loss that he incurs because his payoff is discounted is not bounded away from zero. Hence for times $\underline{t} < t$ close enough to t, there are dates $t' > t$ such that a concession at t' yields a higher payoff than a concession at $\underline{t}$ to all types, i.e., there is $\delta > 0$ such that, with probabiliy 1, the player, and therefore his group (**Y**), does not concede in $(t - \delta, t]$. But then Lemma 1 implies that in $(t - \delta, t]$ group **X** does not concede either, contradicting that $H_X(.)$ has a jump at t. ■

Remark. Note that this argument does not hold for $t = 0$.

LEMMA 3 (Symmetric Spread). *In every (P)BE(US) satisfying Lemma* 1,

(*i*) *if both* T_A *and* T_B *are positive then they are equal*;

(*ii*) *if* $T_X < \infty$ *and* $H_X(T_X) = 1$ *then* $T_X = 0$.

Proof. (i) follows by Lemma 1. To see that (ii) is true, assume that $T_A < \infty$ and $H_A(T_A) = 1$. If $0 < T_A$ then there is some $T_A > \delta > 0$ such that, almost surely, **B** does not give in (recall the inductive argument in the proof of Lemma 1) in $(T_A - \delta, T_A]$, since the conditional probability that **A** will give in in that interval is one. Then, by Lemma 1, $H_A(t)$ is constant in $(T_A - \delta, T_A]$, contradicting the definition of T_A. Therefore $T_A = 0$. ■

LEMMA 4. (Strict Monotonicity). *In every (P)BE(US) satisfying Lemma* 1, *unless the game ends at zero, the probability that either group concedes is positive in any time interval up to* T_X. *That is,* $H_X(t - \delta) < H_X(t)$ *for* $0 < \delta < t \leq T_X$.

Proof. Assume to the contrary that group **X** will not give in with probability 1 in the interval $(t, t + \delta]$. Consequently, by Lemma 1, group **Y** will not give in either. Let δ^* be the supremum of such δ. By Lemma 3, $\delta^* < \infty$ and for any $\tau > t + \delta^*$ the probability of both groups conceding in $[t + \delta^*, \tau)$ is positive. For τ sufficiently close to $t + \delta^*$ every player prefers its group to concede at any t' in $[t, t + \delta^*/2)$ to conceding at τ, since by Lemma 2 the probability that the other group concedes in $[t + \delta^*, \tau)$ is not bounded away from zero. An inductive argument such as that in the proof of Lemma 1 shows that any deviation by i to t would be followed by all her comrades supposed to concede in $[t + \delta^*, \tau)$, and thus would give a group concession at some $t' < t + \delta^*$. This is a contradiction. ■

LEMMA 5 (Independence of Comrades). *In every (P)BE(US) satisfying Lemma* 1, *strategies are independent of the actions of the rest of the players. Moreover, these strategies correspond to the ones the players would play if they were the dictator of their team. That is,* $\sigma_i(s, h_t) \equiv \sigma(s, t)$.

Proof. Fix the strategies of players in **Y**. Let $t_i^*(s)$ be the optimal time of concession of Player i playing as the dictator of **X**. Take any partial history, h_t. Note that, by assumption, Player i only knows about the past actions of group **Y** that not enough of them have conceded yet to give her group the victory. If at t, i finds herself to be pivotal then her best response is to concede at $\max(t, t^*)$. If she is not pivotal, then: (i) if $t < t_i^*$ then she should not concede at t, since by Lemmas 3 and 4 the probability that she will become pivotal before t^* is positive; (ii) if $t \geq t_i^*$ then she should concede, since, again by Lemmas 3 and 4, the probability that she will become pivotal before any $t' > t$ is positive and therefore she could unnecessarily delay the agreement. ■

Note the similarity between the above argument and the proof of why in a sealed-bid second price auction (with independent private values) it is a dominant strategy to bid one's valuation truthfully.

LEMMA 6 (Immediate Concessions). (*i*) *If $Q = N$ then in no BE satisfying Lemma* 1 *do both groups concede at* 0 *with positive probability. That is,* $H_A(0) \cdot H_B(0) = 0$.

(*ii*) *If $Q < N$ then in every BE satisfying Lemma* 1, $H_X(0) \cdot H_Y(0) \in \{0, 1\}$.

Proof. (i) Let group **A** concede at 0 with probability $P > 0$. If the comrades of player j (of group **B**) concede later than 0 with probability 1 the lemma holds true. Otherwise, if j concedes at 0 he either gets a payoff of[11] $(b - s)P - s(1 - P)$ or his move is outcome irrelevant (and thus the

[11] Here $0 < b < 1$ is the (exogenous) probability with which group **B** wins if the two groups concede at the same time.

lemma holds), while if he concedes at $\delta > 0$, he gets at least $(1 - s)P - s(1 - P)e^{-\delta}$ or his move is again outcome irrelevant. Since for any s there is some $\delta(s) > 0$ such that the second payoff dominates the former one, he is better off holding out for a while.

(ii) Note that if according to the strategy profile all the players in a group concede at zero then no individual deviation can change the outcome. On the other hand, if there is a chance that the group does not give in at zero then the argument in (i) applies. ■

LEMMA 10 (Differential Equation System). *Every* $(P)BE(US)$ *satisfying Lemma* 1 *is such that for t in* $(0, T_X)$, $\varepsilon(.)$ *and* $\zeta(.)$, *are a solution to* (1C).

Proof. If $\sigma_j(z)$ is in $(0, T_X)$ for some z, then by Lemmas 5 and 7 it must solve the first-order condition $h_A(t) = -[1 - H_A(t)]z$. Let ϕ: $[0, \infty) \to [i_L, i_H]$ and ν: $[0, \infty) \to [j_L, j_H]$, where $\phi(t) = s$ if and only if $s = \sup\{v$ such that $\sigma_i(v) \leq t\}$, and $v(t) = z$ if and only if $z = \sup\{u$ such that $\sigma_j(u) \leq t\}$. Note that by Lemmas 2, 4, and 8, along the equilibrium path $\phi(t) = \varepsilon(t)$ and $\nu(t) = \zeta(t)$. Therefore, the above condition can be rewritten as $h_A(t) = -[1 - H_A(t)]\zeta(t)$. Moreover, by Lemma 8,[12]

$$\begin{aligned} H_A(t) &= F(\phi(t)) + \sum_{S \in S_A} [F(\mathbf{z}(S)) - F(\phi(t))] \\ &= \sum_{S \in S_A} F(\mathbf{z}(\mathrm{S})) - (\#S_A - 1)F(\phi(t)) \end{aligned}$$

since the probability that group **A** gives in by t is equal to the probability that at least $Q + k - N$ members of it concede by that time. Differentiating, we get that

$$h_A(t) = \sum_{S \in S_A} \phi'(t) \sum_{j \in S} F_j(\mathbf{z}(S)) - (\#S_A - 1)\phi'(t) \sum_{n=1}^{N-k} F_n(\phi(t)).$$

Substituting into the first-order condition we obtain the desired equations. It is straightforward to verify that (1C) simplifies to (1B) and (1A) if $N = Q$ and $N = 2$, respectively. ■

REFERENCES

ASHEIM, G. B. (1992). "A Unique Solution to n-person Sequential Bargaining," *Games and Econ. Behav.* **4,** 169–181.

BENNETT, E., AND HOUBA, H. (1992). "Odd Man Out: Bargaining among Three Players," Working Paper 284, Johns Hopkins University.

[12] $\mathbf{z}$ is the same as $\mathbf{x}$ with ϕ in the place of ε.

BINMORE, K. G. (1985). "Bargaining and Coalitions," in *Game Theoretic Models of Bargaining,* (A. Roth, Ed.), pp. 269–304. Cambridge: Cambridge Univ. Press.

CHAE, S., AND YANG, J. (1988). "The Unique Perfect Equilibrium of an *N*-person Bargaining Game," *Econ. Lett.* **28,** 221–223.

CHATTERJEE, K., DUTTA, B., RAY, D., AND SENGUPTA, K. (1993). "A Noncooperative Theory of Coalitional Bargaining," *Rev. Econ. Studies* **60,** 463–477.

HERRERO, M. J. (1985). *A Strategic Bargaining Approach to Market Institutions,* PhD thesis. London School of Economics.

PONSATI, C., AND SÁKOVICS, J. (1995). "The War of Attrition with Incomplete Information," Math. Soc. Sci. **29,** 239–254.

RAUSSER, G., AND SIMON, L. (1991). "A Non-Cooperative Model of Collective Decision Making: A Multilateral Bargaining Approach," mimeo University of California, Berkeley.

RUBINSTEIN, A. (1982). "Perfect Equilibrium in a Bargaining Model," *Econometrica* **50,** 97–109.

STINCHCOMBE, M. B. (1992). "Maximal Strategy Sets for Continuous-Time Game Theory," *J. Econ. Theory* **56,** 235–265.

YANG, J. (1992). "Another *N*-person Bargaining Game with a Unique Perfect Equilibrium," *Econ. Lett.* **38,** 275–277.