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Multiple-Issue Bargaining and Axiomatic Solutions[1]

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Abstract: We study two-person, multiple-issue bargaining problems and identify four procedures by which the bargaining may take place. Drawing on some logic from non-cooperative game theory, we propose axioms which relate the outcomes of the procedures. We also promote a weak monotonicity axiom on solutions, called issue-by-issue monotonicity, which is geared toward multiple-issue bargaining. Our main result concerns the relationship between a sequential bargaining procedure – with the rule that agreements are implemented only after all issues are resolved – and global bargaining (in which all issues are negotiated simultaneously). If a bargaining solution predicts the same outcome with these two procedures, then we say that it satisfies *agenda independence.* We prove that a solution satisfies axioms of efficiency, symmetry, scale invariance, issue-by-issue monotonicity, and agenda independence if and only if it is the Nash solution. This result provides new intuition for Nash's independence of irrelevant alternatives axiom. Among other results, we show that a solution is invariant to all four of the procedures and satisfies efficiency and symmetry if and only if it is the utilitarian solution with equal weights. We comment on the results of other authors who address multiple-issue bargaining.

Bargaining often involves resolving several different issues. In such situations, parties may elect to negotiate over all of the issues at once, or they may choose to discuss each issue separately. One explanation of why this second option might be preferred is that considering multiple issues at once may be too complicated an exercise for the parties. Indeed, the parties may not have all of the relevant information about the various issues, relying instead on a collection of subordinates who possess specialized knowledge. For whatever the reason, bargaining often takes place through a variety of procedures, in which different issues are handled sequentially or by different agents.

Even if the agents are inspired by the same principles (that is, even if the same bargaining solution is assumed), the specific bargaining procedure that frames

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the negotiation may affect the outcome. Furthermore, to the extent that procedures matter in bargaining, the agents may have an incentive to lobby for the ones that strengthen their prospects. In the case of sequential bargaining, for example, the agents may disagree on the agenda to set.

In this paper, we study two-player bargaining problems that involve multiple issues. Our main goal is to develop a clear understanding of how bargaining procedures affect bargaining outcomes and to characterize solutions based on comparisons of the procedures. We discuss four different procedures. The first is *global bargaining*, in which all issues are negotiated at once. The second, called *separate bargaining*, specifies that the issues are negotiated independently. The final two procedures fall under the heading of *sequential bargaining* and are distinguished by their "rules of implementation." With the rule of *simultaneous implementation*, an agreement on an individual issue cannot be implemented until agreement is reached on all issues. With the *independent implementation* rule, an agreement on an individual issue takes effect immediately. Sequential bargaining also involves a determination of the agenda–the order in which issues will be considered. Along the way toward studying the bargaining procedures, we provide a simple analysis of the geometry of multiple-issue bargaining. We also propose a weak monotonicity axiom on bargaining solutions, which we call *issue-by-issue monotonicity*, that is sensitive to the consideration of multiple issues.

Our main result concerns the procedure of simultaneous implementation sequential bargaining. We present an axiom which states that the outcome of this procedure will always coincide with the outcome of global bargaining. The axiom implies that the particular agenda of sequential bargaining does not affect the outcome of the overall negotiations, and thus there will be no conflict among the players regarding the agenda. Hence, we term this axiom *simultaneous implementation agenda independence*. We show that axioms of efficiency, invariance, symmetry, issue-by-issue monotonicity, and simultaneous implementation agenda independence characterize the Nash bargaining solution. Our theorem offers new intuition for Nash's axiom of independence of irrelevant alternatives.

The analogous exercise of comparing independent implementation sequential bargaining and separate bargaining to global bargaining yields a characterization of the utilitarian solution. The equivalence of separate bargaining and global bargaining is not new to this paper; it has been studied extensively in the literature under the names "linearity" and "additivity." For results in this area, we defer to the important work of Myerson (1981), Perles and Maschler (1981), and Peters (1985, 1986). Regarding independent implementation sequential bargaining, we prove that the utilitarian solution is characterized by the axiom of *independent implementation agenda independence*. We also find that the utilitarian solution is the only one that is independent of the specific bargaining procedure used. That is, the utilitarian solution is characterized in that it is the only solution to predict the same outcome under each of the four bargaining procedures.

In the next section we outline the standard bargaining model and the solutions of Nash (1950) and Kalai and Smorodinsky (1975). In section 2 we explore the geometry of multiple-issues and provide results and intuition that are used throughout the paper. Section 3 motivates, defines, and characterizes our monotonicity axiom. In section 4 we identify the four bargaining procedures and discuss their predictions based on generic solutions. Section 5 contains our main result on agenda independence and the Nash solution and section 6 contains our results on equivalences of procedures ad the utilitarian solution. In section 7 we comment on the step-by-step axiom of Kalai (1977) and Myerson (1977). There we also discuss other related work and conclude the paper.

1 Two-Person Bargaining Problems

A two-person bargaining problem is described by (i) a bargaining set $S \subset \mathbf{R}^2$ which consists of all utility pairs that the players can achieve by reaching a binding agreement, and (ii) a pair of disagreement utilities $d \in \mathbf{R}^2$ that results if the players fail to agree. For simplicity, and without loss of generality, assume that $d = (0, 0)$ so that the bargaining problem is characterized by its bargaining set. Therefore, take $S \subset \mathbf{R}^2_+$ (the non-negative quadrant). Assume as well that S is compact, convex, and comprehensive.[2] Let $\mathbf{B}'$ be the set of all such bargaining sets which contain at least one point $v > 0$ and let $\mathbf{B}$ comprise the bargaining sets which contain at least one point $v \gg 0$.[3] (Axiomatic solutions are defined over $\mathbf{B}$, but herein we will also need to deal with some sets in $\mathbf{B}'$. Unless otherwise noted, we shall always refer to sets in $\mathbf{B}$.) A utility pair $v \in S$ is said to be efficient in S if there is no $u \in S$ such that $u \gg v$. For a bargaining set S, let $eff(S)$ be the set of efficient utility profiles in S.

A bargaining *solution* is a function $f: \mathbf{B} \to \mathbf{R}^2$ such that $f(S) \in S$ for all $S \in B$. That is, a solution selects a pair of feasible utilities for each bargaining problem. For any $S \in \mathbf{B}$ and $a \in \mathbf{R}^2_{++}$ (that is, $a \gg 0$), let $aS \equiv \{(a_1 x_1, a_2 x_2) | x \in S\}$ be the scale transformation of S. Consider the following axioms to which a solution may conform.

Efficiency (EFF): $f(S) \in eff(S)$ for all $S \in \mathbf{B}$.
Symmetry (SYM): If $S \in \mathbf{B}$ is symmetric then $f_1(S) = f_2(S)$.[4]
Invariance (INV): $f(aS) = af(S)$ for all $a \in \mathbf{R}^2_{++}$ and $S \in \mathbf{B}$.

Efficiency demands that no possible gains from cooperation are lost. Symmetry requires that the solution treat players identically if the bargaining set makes them indistinguishable. Invariance mandates that the solution not depend on the specific von Neumann-Morgenstern representation of the preferences of the players.

[2] The set S is comprehensive if for all $u, v \in \mathbf{R}^2_+$ such that $0 \le v \le u$, $u \in S$ implies that $v \in S$.
[3] For $x, y \in \mathbf{R}^2$, $x \gg y$ means that $x_1 > y_1$ and $x_2 > y_2$; $x > y$ means that $x_1 \ge y_1$ and $x_2 \ge y_2$, with at least one of the inequalities strict.
[4] The set S is symmetric if $(x_1, x_2) \in S$ implies that $(x_2, x_1) \in S$.

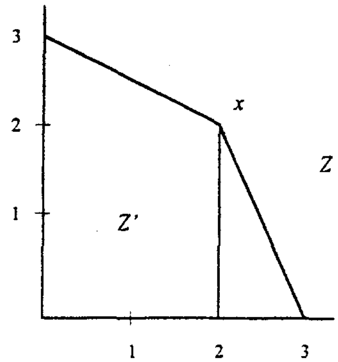

Fig. 1. Luce and Raiffa's example

The *Nash Bargaining Solution* (Nash 1950) is the solution $N:\mathbf{B}\rightarrow\mathbf{R}^2$ in which $N_1(S)N_2(S)=\max\{v_1v_2|(v_1,v_2)\in S\}$. That is, the Nash solution picks the vector in S that maximizes the product of the players' utilities. It is well-known that a solution f is the Nash solution if and only if f satisfies EFF, SYM, INV, and the following axiom:

Independence of Irrelevant Alternatives (IIA): For $S,Z\in\mathbf{B}$, if $Z\subset S$ and $f(S)\in Z$ then $f(Z)=f(S)$.

Nash's axiom of the independence of irrelevant alternatives has been the source of great controversy. To many theorists, it is to strong a requirement of collective rationality and it lacks intuitive support. Much of the criticism is based on the example of Luce ad Raiffa (1957) that is pictured in figure 1. Consider the bargaining sets Z and Z' and note that Z is symmetric. Any solution that satisfies efficiency and symmetry must select point x from the bargaining set Z. If the solution also satisfies axiom IIA then it must also select x from the bargaining set Z'. But Z' is obtained from Z by eliminating all of the alternatives that player 1 prefers to x. Surely, many authors have argued, player 1 ought not fare as well under Z' as she does under the bargaining set Z. This lack of "monotonicity" is seen as an unattractive feature of the Nash Bargaining Solution.

In accordance with this criticism, Kalai and Smorodinsky (1975) advance a monotonicity requirement as an alternative to the IIA axiom. For any bargaining set Z, let $m_iZ\equiv\max\{z_i|z\in Z\}$ be player i's maximum feasible utility, for $i=1,2$.

Individual Monotonicity (IMON): For $Z,S\in\mathbf{B}$, if $Z\subset S$ and $m_iZ=m_iS$ for some some $i=1,2$, then $f_j(Z)\leq f_j(S)$, where j is player i's opponent.

Kalai and Smorodinsky (1975) characterize the unique solution that satisfies EFF, SYM, INV, and IMON. For a bargaining set S, the solution selects the maximal point in S on the line that joins the disagreement point $(0,0)$ with the point (m_1S, m_2S).

2 The Geometry of Multiple Issues

A bargaining problem may involve several issues, in which case the bargaining set represents utility pairs that are feasible through some specification of how individual issues are decided. Consider that each individual issue is described by its own bargaining set and assume that preferences are additive over issues.[5] That is, an n-issue bargaining problem is described by a collection of sets $X^1, X^2, \ldots, X^n \in \mathbf{B}'$ in which the overall set of feasible utility pairs is given by $Z = X^1 + X^2 + \cdots + X^n \equiv \{x^1 + x^2 + \cdots + x^n | x^1 \in X^1, x^2 \in X^2, \ldots, x^n \in X^n\}$. We allow the sets of alternatives for individual issues to be in $\mathbf{B}'$ as long as the sum, which defines the *global* bargaining problem, is in $\mathbf{B}$ (contans some point $v \gg 0$).

Note that the model and solutions discussed in the last section cover multiple-issue bargaining in that they apply to the set Z directly. As we show below, however, one can gain additional insight by studying multiple-issue bargaining more closely. For simplicity, we focus attention on bargaining over two issues, although it will be obvious that our discussion and results extend directly to the general case of a finite number of issues. A few results about the sum of bargaining sets will be useful later and will provide helpful intuition on the nature of multiple-issue bargaining. Our decomposition analysis is basically the same as that of Perles and Maschler (1981).

For any set $S \in \mathbf{B}'$ and any utility $x_1 \in [0, m_1 S]$ for player 1, let $d_+ S(x_1)$ be the slope of the boundary of S *from the right* at x_1. That is, if we regard *eff*(S) as a function of $x_1 \in [0, m_1 S)$ then $d_+ S(x_1)$ is its right derivative at x_1; we adopt the convention of defining $d_+ S(m_1 S) \equiv -\infty$. Correspondingly, let $d_- S(x_1)$ be the slope of *eff*(S) *from the left*. That is, for all $x_1 \in (0, m_1 S]$, $d_- S(x_1)$ is the left derivative at x_1, and $d_- S(0) \equiv 0$.[6] Note that the properties of convexity, compactness, and comprehensiveness are closed under finite addition, so that for $X, Y \in \mathbf{B}', X + Y \in \mathbf{B}'$ as well.

Lemma 1: Suppose $X, Y \in \mathbf{B}'$ and $Z = X + Y \in \mathbf{B}$. Take any $x \in X$ and $y \in Y$ such that $x + y \in \mathrm{eff}(Z)$. Then $x \in \mathrm{eff}(X)$, $y \in \mathrm{eff}(Y)$, and there is an $r \in \mathbf{R} \cup \{-\infty\}$ such that $d_+ X(x_1) \leq r \leq d_- X(x_1)$, $d_+ Y(y_1) \leq r \leq d_- Y(y_1)$, and $d_+ Z(x_1 + y_1) \leq r \leq d_- Z(x_1 + y_1)$. Furthermore, $d_+ Z(x_1 + y_1) \geq d_+ X(x_1), d_+ Y(y_1)$ and $d_- Z(x_1 + y_1) \leq d_- X(x_1), d_- Y(y_1)$.

Proof: simple and left to the reader.

The lemma basically states that for $x + y$ on the efficient frontier of Z, with $x \in X$ and $y \in Y$, the slope of X at x, the slope of Y at y, and the slope of Z at $x + y$ are the same. Furthermore, the boundary of Z is in general less concave than the boundaries of X and Y.

[5] A discussion of this class of preferences may be found in Peters (1985).

[6] Convexity of S implies that these right and left derivatives are well-defined.

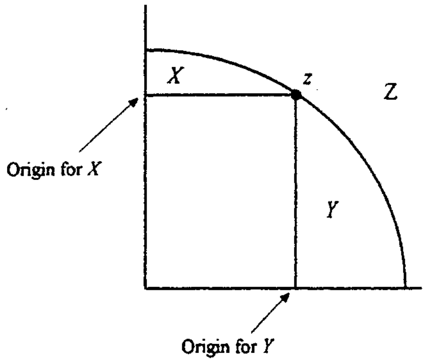

Fig. 2. Decomposing Z along the boundary

One way of examining a multiple-issue bargaining problem is to describe the individual sets that characterize the separate issues and then add them together to find the global bargaining set. Alternatively, we might be given some set Z that defines the overall bargaining problem, with the recognition that Z may represent the sum of several issues. We then need to know the various ways in which Z can be decomposed into two (or more) issues. That is, given $Z \in \mathbf{B}$, what are all the possible sets $X \in \mathbf{B}'$ and $Y \in \mathbf{B}'$ such that $X + Y = Z$?

To answer this question, begin by forming two sets from the boundary of Z by drawing vertical and horizontal lines from any point z on Z's efficient frontier. This procedure is demonstrated in figure 2. Define $g_1(z, Z) \equiv \max\{z_1' | (z_1', z_2) \in Z\} - z_1$ and $g_2(z, Z) \equiv \max\{z_2' | (z_1, z_2') \in Z\} - z_2$. Note that $g_i(z, Z) = 0$ unless $\mathit{eff}(Z)$ is vertical or horizontal at z. Let $X \equiv [Z - (g_1(z, Z), z_2)] \cap \mathbf{R}_+^2$ and $Y \equiv [Z - (z_1, g_2(z, Z))] \cap \mathbf{R}_+^2$. (For any set $W \subset \mathbf{R}^2$ and a vector $\omega \in \mathbf{R}^2$, $W - \omega \equiv \{(x_1 - a_1, x_2 - a_2) | x \in W\}$ is the point-by-point subtraction.) In words, X is the set designated by the area inside Z and above the horizontal line, with the x-axis shifted to the horizontal line. The set Y is defined analogously. These sets are such that $X + Y = Z$:

Lemma 2: Take any $Z \in \mathbf{B}$, any $z \in \mathrm{eff}(Z)$, and let $X \equiv [Z - (g_1(z, Z), z_2)] \cap \mathbf{R}_+^2$ and $Y \equiv [Z - (z_1, g_2(z, Z))] \cap \mathbf{R}_+^2$. Then $X + Y = Z$.

Proof: in appendix A.

By induction, lemma 2 implies that we can partition the efficient boundary of Z into any finite number of segments, forming sets which sum to Z. (See figure 3.) Call such a decomposition of Z a *boundary partition.* Suppose we have a boundary partition of Z which consists of n sets, $W^1, W^2, \ldots, W^n$ (all of which will be members of $\mathbf{B}'$). We can further partition $K = \{1, 2, \ldots, n\}$ into two sets, K' and K'', and then define $X \equiv \sum_{k \in K'} W^k$ and $Y \equiv \sum_{k \in K''} W^k$. Assuming that X and Y are not trivial, this process forms two bargaining sets, $X \in \mathbf{B}'$ and $Y \in \mathbf{B}'$, such that $X + Y = Z$. Therefore, any sets formed from a boundary partition of Z sum to Z. The next lemma shows

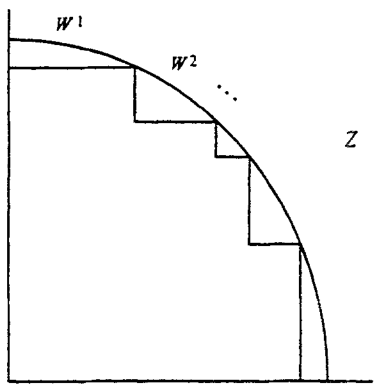

Fig. 3. Boundary partition

that, in a sense, the only sets that sum to Z are those formed from boundary partitions of Z.

For any set $Z \in \mathbf{B}$, let $P(Z)$ be the set of pairs (X, Y) such that X and Y are formed from a boundary partition of Z as described above. Let $\bar{P}(Z)$ be the closure of $P(Z)$ with the Hausdorff metric.

Lemma 3: Take any set $Z \in \mathbf{B}$ and any two sets $X, Y \in \mathbf{B}'$. Then $X + Y = Z$ if and only if $(X, Y) \in \bar{P}(Z)$.

Proof: in appendix A.

Lemma 3 characterizes all of the ways a set may be decomposed. It is helpful in building one's intuition regarding the sum of bargaining sets and it is necessary to prove our next result.

3 Monotonicity and Multiple Issues

It would seem as though a reasonable bargaining solution should exhibit a measure of monotonicity, like that embodied by the IMON axiom of Kalai and Smorodinsky (1975). However, there is a sense in which the IMON axiom is too strong a requirement for bargaining that involves multiple issues. To illustrate, suppose that we have sets Z, X, and Y such that $Z = X + Y$ (X and Y are separate issues that, combined, yield the global bargaining problem defined by Z). Now take another set $Z' \subset Z$ such that $m_2 Z' = m_2 Z$. The IMON axiom would require that player 1 fare no better with Z' as she does with Z. However, it may not be obvious that player 1 is disadvantaged under Z' when we consider the separate underlying issues. Suppose these issues are represented by the sets X' and Y', where $X' + Y' = Z'$. It is possible that $m_2 X' = m_2 X$, $m_2 Y' = m_2 Y$, and $X' \subset X$, yet $Y \subset Y'$.

That is, player 1's opportunities are reduced in the first issue, but *expand* in the second issue. To the extent that the bargaining set is an abstraction, which in general aggregates a variety of issues, a weak monotonicity notion should be consistent with comparisons that are made on an issue-by-issue basis. This is especially important considering that, in reality, issues are often addressed separately.

We propose a weaker version of the IMON axiom that is sensitive to comparisons made on an issue-by-issue basis. A few preliminary definitions will be helpful. Note that we will use the convention that given a player i, j denotes i's opponent. Take two sets $X, X' \in \mathbf{B}'$. We shall say that X' is an *i-reduction* of X if $m_j X = m_j X'$ and $X' \subset X$. Then for any sets $Z, Z' \in \mathbf{B}'$, we shall say that Z' is an *strong i-reduction* of Z if and only if for all $X, Y \in \mathbf{B}'$ such that $X + Y = Z$, there are sets X', $Y' \in \mathbf{B}'$ for which $X' + Y' = Z'$, X' is an i-reduction of X, and Y' is an i-reduction of Y. That is, Z' is a strong i-reduction of Z if and only if for each decomposition of Z into two issues, there is a corresponding decomposition of Z' which represents an i-reduction on both issues.

There is a simple mathematical characterization of strong i-reductions, a characterization established by the next lemma. For any set $Z \in \mathbf{B}'$ and $x_2 \in [0, m_2 Z]$, let $p_+ Z(x_2)$ be the slope of *eff*(Z) from the right at the point x_2. (This is just like $d_+ Z$, only as a function of player 2's utility as opposed to player 1's.)

Lemma 4: Take any two sets $Z, Z' \in \mathbf{B}'$. Then Z' is a strong 1-reduction of Z if and only if Z' is a 1-reduction of Z and $p_+ Z'(x_2) \leq p_+ Z(x_2)$ *for* all $x_2 \in [0, m_2 Z]$. Likewise, Z' is a strong 2-reduction of Z if and only if Z' is a 2-reduction of Z and $d_+ Z'(x_1) \geq d_+ Z(x_1)$ for all $x_1 \in [0, m_1 Z]$.

Proof: in appendix A.

As this lemma demonstrates, strong i-reductions involve monotone changes in the slope of the boundary of a given set, retracting the set in one direction. For example, a set Z' is a strong 2-reduction of Z if and only if Z' is a 2-reduction of Z and the slope of the boundary of Z' is weakly greater than the slope of the boundary of Z, at each possible utility level of player 1. Some readers may find this slope condition more attractive than our motivation based on multiple issues.

The concept of a strong i-reduction is the basis for a weak monotonicity axiom.

Issue-by-Issue Individual Monotonicity (IIM): For Z, $Z' \in \mathbf{B}$, if Z' is a strong i-reduction of Z then $f_i(Z') \leq f_i(Z)$.

Clearly, IIM is less stringent a requirement than is IMON; the latter implies the former. To compare IIM with another monotonicity axiom, take the "twisting" axiom of Thomson and Myerson (1980).[7] Roughly, twisting demands that, for a bargaining set S, expanding S on one side of $f(S)$ (where player i's utility is greater than $f_i(S)$) and retracting S on the other side of $f(S)$ weakly improves player i's

[7] Also see Thomson (1994, forthcoming).

outcome.[8] (Note that the comparison of sets depends on the bargaining solution.) One can show that EFF and twisting imply IIM. (We thank William Thomson for a proof.) In fact, IIM is so weak that it is satisfied by *all* of the major bargaining solutions.

Lemma 5: The following solutions satisfy the IIM axiom: Nash, generalized Nash, Kalai and Smorodinsky, Perles and Maschler, proportional (Kalai 1977), and utilitarian (Myerson 1981).

Proof: omitted.

We have defined the Nash and Kalai and Smorodinsky solutions already, and in sections 5, 6, and 7 we deal with the other solutions mentioned in lemma 5.

Henceforth, we shall take the EFF, SYM, INV, and IIM axioms as fundamental, although for some results we will have to drop the INV axiom. These four axioms – which are satisfied by the Nash, Kalai and Smorodinsky, and Perles and Maschler solutions – are more or less generally accepted. (Of the axioms, INV gets the most criticism and is not satisfied by proportional and utilitarian solutions, which imply utility comparisons.) It is well-known that dispensing with SYM leads to generalizations of many solutions. However, rather than seek the most general form of our results, which would require defining additional axioms and would create a more cluttered paper, we shall concentrate on our specific contributions. Throughout the analysis, we will indicate where the results can be generalized a bit.

4 Procedures for Multiple-Issue Bargaining

Bargaining over multiple issues may proceed in a variety of ways, with the outcome not necessarily independent of the framework within which the negotiation takes place. A brief survey of the different bargaining procedures will be helpful, in fact essential, in understanding and interpreting our results and the results of others. We continue to focus on bargaining problems in which there are two issues to be resolved. The two issues are represented by the bargaining sets X and Y. The overall, or global, bargaining problem thus corresponds to the set $Z = X + Y$.

One way for the agents to proceed is to directly tackle the global problem Z, in which both issues are addressed at once. For example, by analogy to non-cooperative theory, an offer from one agent to the other would involve a specification of how both X and Y are to be resolved. Another procedure for bargaining might specify that negotiations over X and Y are totally separate and independent,

[8] To be precise, twisting requires that for all $S, T \in \mathbf{B}$ and each i, if (1) $f(S)$ is on the boundary of T, (2) $x \in T \setminus S$ implies that $x_j \leq f_j(S)$, and (3) $x \in S$ and $x_j \leq f_j(S)$ implies that $x \in T$, then either $f_i(T) > f_i(S)$ or $f_i(T) = f_i(S)$ and $f_j(T) \leq f_j(S)$.

with each having no effect on the other. Such would be the case if each of the two parties employed two agents, one in charge of X and the other in charge of Y. For example, between two countries there may be two issues at stake, each of which is resolved – through bargaining – by representatives from the countries who care only about their narrow issue. We call such an arrangement *separate bargaining.*

In perhaps the most natural way for bargaining to proceed, the parties take the issues one at a time. For instance, they may negotiate over the issue X and then, after reaching an agreement on X, move to negotiate Y. This we call *sequential bargaining.* A central premise of this procedure is that the parties may not negotiate the second issue until the first is resolved. Several forms of sequential bargaining need to be distinguished, and we do so along two lines: the agenda and the rule of implementation. The agenda specifies which of the two issues will be bargained over first. The rule of implementation specifies when and whether an agreement on an individual issue goes into effect. We consider two such rules. The rule of *independent implementation* states that an agreement on the first issue goes into effect immediately (before negotiation begins on the second issue), while the rule of *simultaneous implementation* does not allow an agreement on the first issue to take effect until subsequent agreement is reached on the second issue. These distinctions are critical.

To summarize, we have identified four procedures: global bargaining, separate bargaining, independent implementation sequential bargaining, and simultaneous implementation sequential bargaining. (The last two procedures are further partitioned according to the agenda.) Given a solution f, we can determine the outcome of the bargaining that is implied by each of these procedures. Obviously, the prediction for global bargaining is $f(X + Y)$ and the prediction for separate bargaining is $f(X) + f(Y)$. Sequential bargaining is more involved. We determine the solutions for the two sequential procedures with the help of some non-cooperative game theory logic.

For any set $W \subset \mathbf{R}^2_+$, let $comp(W) \equiv \{x \in \mathbf{R}^2_+ | x \leq y \text{ for some } y \in W\}$ be the comprehensive extension of W. Also, for any $W \subset \mathbf{R}^2_+$, let $chcp(W)$ denote the convex hull of $comp(W)$. Consider the rule of independent implementation and take the case in which the agenda specifies that X be bargained over first, followed by Y. Since the agreement made on X is implemented immediately, it will not affect the bargaining in the second stage of the procedure. To see this, suppose that $x \in X$ is the agreement in the first stage of the procedure. Then, in the second stage, the relevant bargaining set is $x + Y$ with x as the disagreement point (since x has taken effect and cannot be altered). Normalizing the disagreement point to $(0, 0)$ makes Y the bargaining set that the parties face in the second stage. Thus, regardless of the agreement in the first stage, $f(Y)$ designates the resolution of the negotiation over Y. In the first stage of the procedure, then, the players realize that $f(Y)$ will be the agreement on Y, but this will not occur until they resolve the issue X. Therefore, in the first stage, the agents actually bargain over the set $f(Y) + X$ (with disagreement point $(0, 0)$, since if no agreement is reached then the players cannot advance to the second stage). Assuming disposal is possible, we take the comprehensive version, $comp(f(Y) + X)$. Thus, in the independent implementation bargaining problem with X negotiated first, the solution f predicts the agreement $f(comp(f(Y)+X))$.

Now consider sequential bargaining with the simultaneous implementation rule. Again, take the case in which X is negotiated first, followed by Y. Suppose that $x \in X$ is the agreement reached in the first stage. Then in the second stage, the parties ostensibly negotiate over Y, but *in reality* negotiate over $comp(x+Y)$. This is because an agreement in the second stage not only leads to the implementation of the issue Y but to the implementation of x as well. If no agreement is reached in the second stage, then x cannot be implemented, so the disagreement point in the second stage is $(0,0)$. Thus, given x, negotiation over Y leads to the agreement $f(comp(x+Y))$. Stepping back to the first stage, notice that the relevant bargaining set is

$$T(X, Y; f) \equiv chcp\{f(comp(x+Y)) \mid x \in X\},$$

since an agreement $x \in X$ in the first stage inevitably leads to the payoff profile $f(comp(x+Y))$ via negotiation over Y.[9] Thus, for the sequential bargaining problem with X negotiated first and with the rule of simultaneous implementation, the solution f predicts the agreement $f(T(X, Y; f))$.

5 Simultaneous Implementation and the Nash Solution

Global bargaining seems to be a fair procedure, since all issues are "on the table" at once. However, in practice multiple-issue bargaining rarely takes this form. It is therefore important to understand how specific bargaining procedures affect the outcome. Indeed, bargaining solutions may be characterized by the relationship between their predictions under the various regimes. In this section we present our main result, which concerns simultaneous implementation sequential bargaining and the Nash solution. The following section contains results for the other procedures.

Consider sequential bargaining with the rule of simultaneous implementation. One might argue that rational agents should reach the same (overall) agreement with this procedure as they would if they bargained over both issues at once. This is because of simultaneous implementation and that the agents are forward-looking. Simultaneous implementation guarantees that each party can block implementation of both issues, even after agreement is reached on the first, by withholding agreement on the issue that is negotiated second. Thus, agreement on the second issue requires that both agents are satisfied with their combined shares. Furthermore, since the agents understand how an agreement on the first issue "ties their

[9] Assuming the availability of a randomization device to create lotteries, we have used the convex hull in the definition of T, which ensures that $T \in \mathbf{B}$. This is standard practice in the literature.

hands" in the second stage of negotiations, they would not be willing to make a first-stage agreement that left them in a poor bargaining position later.

If this logic holds, then the agenda in sequential bargaining will not affect the overall agreement that is reached. Bargaining over X first yields the same agreement as if Y were first negotiated, and both are equivalent to global bargaining. This equivalence is captured by the following axiom:

Simultaneous Implementation Agenda Independence (SIAI): For all $X, Y \in \mathbf{B}'$ such that $X + Y \in \mathbf{B}, f(X + Y) = f(T(X, Y; f))$.

The axiom states that global bargaining and sequential bargaining with simultaneous implementation yield the same agreement.

Our main result establishes that, given the fundamental axioms of EFF, SYM, INV, and IIM, the SIAI axiom characterizes the Nash solution. In comparison to Nash's characterization, one may replace IIA with IIM and SIAI.

Theorem 1: A bargaining solution satisfies EFF, SYM, INV, IIM, and SIAI if and only if it is the Nash solution.

We prove the result below.

This theorem provides a useful reinterpretation of Nash's axiom of the independence of irrelevant alternatives that is based on the recognition that a bargaining problem will generally consist of sequential negotiation over several issues. Any agreement in the first stage of negotiation implies that a subset of the global bargaining set will be bargained over in the second stage. The SIAI axiom requires that the parties reach an agreement in the first stage that induces such a reduction in the second stage such that the second-stage agreement will match the agreement from global bargaining. Intuitively, this is substantially weaker than the demand of IIA, but of a similar flavor. Of course, IIA requires equivalence of the agreement for any reduction of the bargaining set that contains its agreement point.

We conjecture that Theorem 1 can be generalized by dropping the SYM axiom and replacing "Nash solution" with "generalized Nash solution." We can prove that EFF, IIM, and SIAI imply IIA for bargaining sets that are strictly convex, but we haven't been able to extend the implication to all bargaining sets. With this resolved, one could use the results of de Koster, Peters, Tijs, and Wakker (1983) to prove the more general result. These authors find that a solution satisfies strict Pareto optimality, invariance, and independence of irrelevant alternatives if and only if it is a generalized Nash solution or is dictatorial. Removing the INV axiom opens the door to even more solutions. Our result in the next section on solutions that are invariant to the bargaining procedure demonstrates that utilitarian solutions are included in this case.

Proof of Theorem 1: First we must show that the Nash solution satisfies the five axioms. We know already that EFF, SYM, INV, and IIM are satisfied, so we simply must demonstrate that the Nash solution satisfies SIAI. Take $X, Y \in \mathbf{B}'$ such

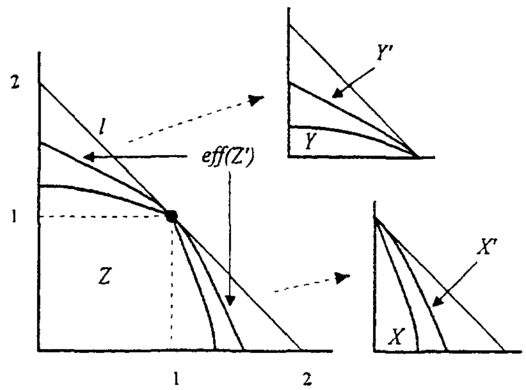

Fig. 4. Construction of Z', X, X', Y, and Y'

that $X + Y \in \mathbf{B}$ and recall that N denotes the Nash solution. Note that $comp(x + Y) \subset X + Y$ for all $x \in X$. Let $x^* \in X$ and $y^* \in Y$ be such that $x^* + y^* = N(X + Y)$. Note that $x^* + y^* \in comp(x^* + Y)$. Thus IIA implies that $N(comp(x^* + Y)) = N(X + Y)$. Obviously $N(comp(x^* + Y)) \in chcp\{N(comp(x + Y)) | x \in X\} = T(X, Y; N) \subset X + Y$. By IIA again, $N(X + Y) = N(T(X,Y; N))$, proving that N satisfies SIAI.

Next we must show that if a solution f satisfies the five axioms, then it must be the Nash solution. We do this in two steps. First we prove the result for sets which are strictly convex at the Nash solution, and then we extend the result to all other sets in $\mathbf{B}$. By INV, we can restrict attention to sets $Z \in \mathbf{B}$ such that $N(Z) = (1, 1)$. Note that for such a set Z, $d_+(1) \leq -1$ and $d_-Z(1) \geq -1$.

Step 1: Suppose Z is strictly convex at $(1, 1)$. Then the line l defined by $u_2 = 2 - u_1$ is a line of support for Z and it intersects Z uniquely at $(1, 1)$. We can then find a set $Z' \in \mathbf{B}$ with the following properties: $Z \subset Z', Z'$ is symmetric and strictly convex at $(1, 1)$, $eff(Z) \cap eff(Z') = \{(1, 1)\}$, $d_+Z(x_1) \geq d_+Z'(x_1)$ for all $x_1 \in [0, 1)$, and $p_+Z(x_2) \leq p_+Z'(x_2)$, for all $x_2 \in [0, 1)$. That is, Z' is a symmetric set that is strictly convex at $(1, 1)$, constructed so that $eff(Z')$ lies between $eff(Z)$ and the line l, with tangency only at the point $(1, 1)$. (See figure 4.)

Let $Y \equiv [Z - (0, 1)] \cap \mathbf{R}^2_+$, $X \equiv [Z - (1, 0)] \cap \mathbf{R}^2_+$ $Y' \equiv [Z' - (0, 1)] \cap \mathbf{R}^2_+$, and $X' \equiv [Z' - (1, 0)] \cap \mathbf{R}^2_+$. By construction, $X + Y = Z$ and $X' + Y' = Z'$, Y is a strong 2-reduction of Y', and X is a strong 1-reduction of X'. Also, the set $X + Y'$ is strictly convex at the efficient point $(1, 1)$. Let $z = f(X + Y')$. By EFF, $z \in eff(X + Y')$. By EFF and SYM, $f(X' + Y') = (1, 1)$.

We will show that $z_1 \geq 1$. To do so, suppose, $z_1 < 1$ and we shall derive a contradiction. Compare the two sequential bargaining problems (with simultaneous implementation) defined by $X + Y'$ and $X' + Y'$, where X (respectively, X') is negotiated first. By SIAI, $f(T(X, Y'; f)) = z$ and $f(T(X', Y'; f)) = (1, 1)$. Note that by the convexity property of $X + Y'$ (at $(1, 1)$), if for $x \in X$ and $y \in Y$,

$x + y \in eff(X + Y')$ and $x_1 + y_1 < 1$, it must be that $x = (0, 1)$. This implies that $(0, 1)$ must be the agreement on X in the first stage of the sequential bargaining problem over $X + Y'$ in which X is negotiated first (with simultaneous implementation). A similar argument establishes that $(0, 1)$ must be the first-stage agreement on X' in the sequential bargaining problem over $X' + Y'$ in which X' is negotiated first. Considering the second stage of these bargaining games, it must be that $f(comp((0, 1) + Y')) = z$ (in the $X + Y'$ case) and $f(comp((0, 1) + Y')) = (1, 1)$ (in the $X' + Y'$ case), which is a contradiction. Therefore $z_1 \geq 1$.

Note that, by construction, $X + Y'$ is a strong 1-reduction of $X' + Y'$, so IIM implies that $z_1 \leq 1$. Therefore, it must be that $z = (1, 1)$. Analogously, since all of the relevant sets are strictly convex at $(1, 1)$, we can show that $f(X + Y) = (1, 1)$, and so $f(Z) = N(Z)$.

Step 2: Now take the case in which Z is not strictly convex at the Nash point $(1, 1)$. Assume, for the purpose of deriving a contradiction, that $f(Z) \equiv z \neq (1, 1)$. Without loss of generality, suppose that $z_1 < 1$. First notice that it cannot be that there exists a point $s \in eff(Z)$ such that $s_2 = z_2$ and $s_1 > z_1$, as would be true if z were in the interior of a horizontal segment of $eff(Z)$. (If this were true then we could let $S \equiv \{x | 0 \leq x \leq s\}$. The axioms SYM, EFF, and INV then imply that $f(S) = s$. But it is easy to see that S is a strong 1-reduction of Z, so IIM implies that $s_1 \leq z_1$, which is a contradiction.)

Notice that we can find a set Z' with the following properties: $Z' \subset Z$, $eff(Z')$ coincides with $eff(Z)$ at all points in which player 1's utility is less than or equal to z_1, $p_+Z'(x_2) \leq p_+Z(x_2)$ for all $x_2 \in [0, z_2]$, Z' is strictly convex at $z' \equiv N(Z')$, and $z'_1 > z_1$. By step 1 above, $f(Z') = N(Z') = z'$. But by construction, Z' is a strong 1-reduction of Z, and so IIM implies that $z'_1 \leq z_1$, which is a contradiction. Therefore, we conclude that $f(Z) = (1, 1) = N(Z)$. This completes step 2. *Q.E.D.*

6 Other Results

We have characterized the Nash solution based on a comparison of global bargaining and simultaneous implementation sequential bargaining. Relating global bargaining to the other two procedures characterizes another solution. Take the following axiom:

Separate/Global Equivalence (SGE): $f(X + Y) = f(X) + f(Y)$ for all $X, Y \in \mathbf{B}$.

The SGE axiom has been studied by Myerson (1981) in a social choice context, under the name *linearity*.[10] He shows that linearity and efficiency imply that

[10] Myerson's linearity condition is actually $f(\alpha X + (1 - \alpha)Y) = \alpha f(X) + (1 - \alpha)f(Y)$. He studies social choice before and after the resolution of some uncertainty. Chun (1988) proposes a weaker version of linearity (binding only for problems S and Z such that $f(S) = f(Z)$) and with EFF, SYM, and INV characterizes the Nash Solution.

a choice function is *utilitarian*, in that it maximizes the sum of a weighting of the players' utilities. For our purposes, it will be most convenient to add the symmetry axiom, yielding an exact characterization of the solution.

The *symmetric utilitarian solution* SU: $\mathbf{B} \to \mathbf{R}^2_+$ is defined as follows. For $Z \in \mathbf{B}$, let $M(Z) \equiv \{x \in Z \mid \text{for all } y \in Z, x_1 + x_2 \geq y_1 + y_2\}$, which is a linear, closed interval in $\mathbf{R}^2_+$ (which may contain just one point). Let v and v' be the extreme points (endpoints) of $M(Z)$. (If $M(Z)$ contains only one point, then $v = v'$ is this point.) Then $SU(Z) \equiv ((v_1 + v'_1)/2, (v_2 + v'_2)/2)$. Here is a more specific version of Myerson's result, which we state without proof:

Theorem 2: A solution satisfies EFF, SYM, and SGE if and only if it is symmetric utilitarian.

Remember that utilitarian solutions satisfy our IIM axiom. Also note that we have dropped the invariance axiom from our list, which would imply nonexistence of a solution.

Several authors have characterized solutions based on another axiom, which implies a weaker relationship between global and separate bargaining:

Super-Additivity (SA): $f(X + Y) \geq f(X) + f(Y)$, for all $X, Y \in \mathbf{B}$.

Super-additivity implies that the agents always weakly prefer to bargain over the global bargaining set than take issues separately and independently. Perles and Maschler (1981) characterize the unique solution that satisfies EFF, INV, SYM, SA, as well as a requirement of continuity.[11] Peters (1986) proves that a solution satisfies EFF, SA, individual rationality (which is superfluous in the present context), and a homogeneity condition if and only if it is proportional. He also characterizes proportional solutions with an axiom that is weaker than super-additivity.

Consider next the relationship between global bargaining and independent implementation sequential bargaining.

Independent Implementation Agenda Independence (IIAI): For all $X, Y \in \mathbf{B}'$ such that $X + Y \in \mathbf{B}$, $f(X + Y) = f(comp(f(Y) + X))$.

This axiom requires that independent implementation sequential bargaining yields the same outcome as does global bargaining. Thus, like SIAI, it implies that the agenda does not affect the outcome. Like the SGE axiom, however, IIAI characterizes utilitarian solutions.

Theorem 3: A solution satisfies EFF, SYM, and IIAI if and only if it is symmetric utilitarian.

[11] Their interpretation of separate issues are bargaining sets which emerge after the resolution of some uncertainty. The SA axiom, in their context, implies that agents are always willing to bargain before the uncertainty is resolved.

Proof: in appendix B.

As with theorem 2, a family of utilitarian solutions is characterized when one removes the SYM axiom.

Our final result addresses whether there is a solution that is invariant to all four of the bargaining procedures. For multiple-issue bargaining problems, such a solution predicts the same agreement regardless of the procedure by which the players negotiate. In fact, the utilitarian solution has this property.

Theorem 4: A solution satisfies EFF, SYM, SGE, IIAI, and SIAI if and only if it is symmetric utilitarian.

Proof: From theorems 2 and 3, we know that a solution is symmetric utilitarian if and only if it satisfies EFF, SYM, SGE, and IIAI. Thus, we only need to show that the symmetric utilitarian solution satisfies the SIAI axiom. Recall that SU denotes the symmetric utilitarian solution. It is not difficult to see that $SU(comp(x + Y)) = x + SU(Y)$ for any $x \in \mathbf{R}^2_+$. Therefore, $T(X, Y; SU) = chcp(SU(Y) + X) = comp(SU(Y) + X)$. Since SU satisfies IIAI, we know that $SU[comp(SU(Y) + X)] = SU(X + Y)$. This means that $SU(T(X, Y; SU)) = SU(X + Y)$, which completes the proof. *Q.E.D.*

7 Comments and Related Work

In the last section, we cited work in the literature that examines the relationship between global and separate bargaining. As far as we can tell, no work has been done on the relationships between the other bargaining procedures that we have studied here (and focused on). There is, however, at least one other prominent strand of the literature which addresses multiple-issue bargaining and motivates a bargaining axiom through consideration of specific bargaining procedures. On this research we would like to comment.

The papers of Kalai (1977) and Myerson (1977) propose the following axiom:

Step by Step Negotiations (SSN): For all X, $Z \in \mathbf{B}$ such that $X \subset Z$, $f(Z) = f(X) + f((Z - f(X)) \cap \mathbf{R}^2_+)$.

One popular motivation of the SSN axiom runs like this. Suppose agents negotiate an agreement on X and then discover that more utility pairs, corresponding to the set Z, are feasible. ($X \subset Z$). The agents may then re-open the bargaining, but if they fail to reach an agreement on Z then their original agreement on X stands. The SSN axiom is meant to capture the assertion that this procedure yields the same agreement as would have been reached if the agents bargained over Z directly.

Kalai (1977) argues that the axiom addresses multiple-issue bargaining in the sense that X might be one sub-issue of the global bargaining problem Z. That is, we can think of there being two issues, X and Y, with the global bargaining problem defined by $X + Y = Z$. The agents bargain over X first, until an agreement is made. Then negotiations are extended to the set $X + Y$, with the rule that the agreement on X stands if no subsequent agreement is reached. That is, the agents start with the issue X and then, after reaching an agreement on X, consider X and Y together.

In light of the bargaining procedures discussed in section 4, we have two observations regarding the motivation of the SSN axiom. The first concerns bargaining over Z after an agreement has been reached on X. As Kalai pointed out, that the agents bargain over the set $[Z - f(X)] \cap \mathbf{R}^2_+$ in the second stage of the procedure implies that they must be able to alter their agreement on X at this time.[12] But it also implies that the agreement on X is the disagreement point for bargaining over Z in the second stage. (Bargaining over $[Z - f(X)] \cap \mathbf{R}^2_+$ is tantamount to bargaining over Z with disagreement point $f(X)$.) These assumptions seem reasonable if (1) implementation of the agreement on X is postponed until negotiation on Z concludes, and (2) bargaining impasse (perhaps through exogenous forces) during the negotiation on Z implies that the agreement on X is automatically implemented.[13] The assumptions are not reasonable if an agent could block implementation of $f(X)$ by withholding agreement in the second stage.

Our second comment on the SSN axiom concerns what it implies for bargaining over X in the first stage. With the IIAI and SIAI axioms that we propose, the agents understand that Y will be negotiated in the second stage. They understand that bargaining over X is affected by the shadow of the second stage and that an agreement on X may have implications for bargaining over Y. On the other hand, with the SSN axiom, players are assumed to ignore – or not know about – the problem they will face in the second stage. One might argue (as is suggested by Myerson's (1977) interpretation) that the players do not know the nature of Y until the second stage begins, and for this reason reach the agreement $f(X)$ on X in the first stage. Or one might argue that the agents are boundedly rational and don't assess the strategic connection between first- and second-stage bargaining. In the least, SSN implies a very different assumption about the agents' understanding of their situation than is implied by our axioms. Perhaps it would be instructional to study the middle ground.

Before concluding our presentation, we would like to acknowledge the relevant work by others which we have not yet discussed. Fershtman (1990) analyzes a non-cooperative bargaining problem over two issues, where agreements are implemented only after both issues are resolved. In his model of alternating-offers and impatient players, the unique subgame perfect equilibrium depends on the

[12] Otherwise, they would be bargaining over $Y \neq [Z - f(X)] \cap \mathbf{R}^2_+$.

[13] Alternatively, an agreement on X may initiate a flow payoff that can be changed due to renegotiation.

order in which the issues are negotiated. The dependence vanishes as the interval between offers approaches zero. This result is consistent with our main theorem, since the outcome of the alternating-offer (Rubinstein) game converges to the Nash solution as the period length shrinks to zero. Ponsati (1992) presents a two-sided, incomplete information bargaining game involving multiple issues, each of which can be resolved in only two possible ways. Conditions under which there is a unique perfect Bayesian equilibrium are provided. In this equilibrium, the agents usually reach an agreement by trading a favorable solution on some issue with an unfavorable one on another, and some delay is necessary.

Finally, Herrero (1993) analyzes two-issue bargaining from both the axiomatic and non-cooperative viewpoints and considers simultaneous implementation and independent implementation. Her focus is quite different from ours. She studies bargaining problems that are not necessarily convex, taking the non-convex Nash solution and a non-cooperative counterpart as fundamental. She studies how the outcome of bargaining depends on the procedure and shows that the outcome of sequential bargaining depends on the agenda. For convex problems, she demonstrates that the Nash solution satisfies (in our terminology) simultaneous implementation agenda independence. For non-convex problems, there is a potentially acute discrepancy between the outcome of global bargaining and the outcome of sequential bargaining. It may be that, while only a few points are possible solutions of global bargaining, any efficient point can arise as a solution in sequential bargaining.

Our goal in this paper has been to treat rigorously the various procedures by which multiple-issue bargaining may take place. We have offered several axioms that relate the procedures and have suggested a monotonicity axiom that is sensitive to the possibility that a bargaining set may actually be composed of several issues. Our results characterize both the Nash and utilitarian solutions. We believe that further research on the nature of bargaining procedures for multiple issues would be constructive.

A Proofs of Lemmas

Proof of lemma 2: Let $x=(m_1X,0)\in X$ and let $y=(0,m_2Y)\in Y$. Observe that $[x+\mathit{eff}(Y)]\cup[y+\mathit{eff}(X)]=\mathit{eff}(Z)$, by construction, which implies that $Z\subset X+Y$. Now take any point $w\in\mathbf{R}^2$ that lies outside of Z. Since Z is convex, we can find a line l that is tangent to Z at some point $t\in\mathit{eff}(Z)$ such that l separates Z from w. Let $t'\in X$ and $t''\in Y$ be such that $t=t'+t''$. Lemma 1 implies that we can find lines l' and l'' of the same slope as l such that l' is tangent to X at t', l'' is tangent to Y at t'', l' bounds X from above, and l'' bounds Y from above. By construction, $l=l'+l''$, which means that l bounds $X+Y$ from above, proving that $w\notin X+Y$. Therefore, $X+Y=Z$. *Q.E.D.*

Proof of lemma 3: For any set $S \in \mathbf{B}$, let $bS: \mathbf{R} \to \mathbf{R}$ be the function that defines the boundary of S. That is, for $x_1 \in [0, m_1 S]$, $bS(x_1) = \max\{x_2 | (x_1, x_2) \in S\}$. At $m_1 S$ we recognize that the boundary extends from $(m_1 S, 0)$ to $(m_1 S, bS(m_1 S))$. .

First we must show that $(X, Y) \in \bar{P}(Z)$ implies that $X + Y = Z$. We already know that $(X, Y) \in P(Z)$ implies that $X + Y = Z$. Let δ denote the Hausdorff distance. It is easy to see that for any sets $X, X', Y, Y' \in \mathbf{R}^2$, $\delta(X + Y, X' + Y') \leq \delta(X, X') + \delta(Y, Y')$. Take any $(X, Y) \in \bar{P}(Z)$ and let ε be any positive number. We can find a pair $(X', Y') \in P(Z)$ such that $\delta(X, X') < \varepsilon/2$ and $\delta(Y, Y') < \varepsilon/2$. Therefore $\delta(X + Y, Z) = \delta(X + Y, X' + Y') < \varepsilon$. Since ε is arbitrary, it must be that $\delta(X + Y, Z) = 0$. Because these sets are closed, this means that $X + Y = Z$.

Next we must show that if $X, Y \in \mathbf{B}'$ are such that $X + Y = Z$, then $(X, Y) \in \bar{P}(Z)$. Take any $X, Y \in \mathbf{B}'$ such that $X + Y = Z$. We show below, through a series of steps, that there is some $Y' \in \mathbf{B}'$ such that $X + Y' = Z$ and $(X, Y') \in \bar{P}(Z)$. Our proof is for the case in which $bZ(m_1 Z) = 0$ and $d_+ Z(0) < 0$ (hence $bX(m_1 X) = 0$ and $d_+ X(0) < 0$ as well). This ensures that the boundaries of these sets do not have completely vertical or horizontal segments, which simplifies the exposition a bit. The proof is easily extended to cover the case in which $bZ(m_1 Z) = 0$ and/or $d_+ Z(0) = 0$. Given the existence of Y', we have that $X, Y, Y' \in \mathbf{B}'$, $X + Y = Z$, and $X + Y' = Z$. A separating line argument then establishes that $Y = Y'$. Therefore, $(X, Y) \in \bar{P}(Z)$, which completes the proof.

Step 1: Partitioning X. Take any positive integer K and let $\varepsilon = m_1 X / K$. We shall decompose X into K sets by forming a boundary partition that cuts the boundary of X at each of the points $\varepsilon, 2\varepsilon, \ldots, (K-1)\varepsilon$ along the x-axis. (See figure 5.) This forms the boundary partition $\{X^k\}_{k=1}^{K}$. Mathematically, for $k = 1, 2, \ldots, K$,

$$X^k \equiv [X - ((k-1)\varepsilon, bX(k\varepsilon))] \cap \mathbf{R}^2_+.$$

Note that $bX(K\varepsilon) = 0$.

Step 2: Partitioning Z. Now that we have partitioned the boundary of X, we wish to do the same for Z, selecting sets that most closely "match" the sets created by the decomposition of X. Considering first the set X^K, let $W^K \equiv [Z - (m_1 Z - \varepsilon, 0)] \cap \mathbf{R}^2_+$. Next, define the sets $W^1, W^2, \ldots, W^{K-1}$ as follows.

Let $y_1^K \equiv m_1 Z$ and define numbers $y_1^1, y_1^2, \ldots, y_1^{K-1}$ inductively, so that

$$y_1^k = \max\{y_1 | y_1 \leq y_1^{k+1} - \varepsilon, \quad d_+ X(k\varepsilon) \leq d_- Z(y_1)\},$$

for $k = 1, 2, \ldots, K-1$. That is, each y_1^k is at least ε away from its neighbors and the slope of *eff*(Z) at y_1^k is at least as great as the slope of *eff*(X) at $x_1 = k\varepsilon$. Using lemma 1 we can show that $d_- Z(y_1^k) \leq d_- X(k\varepsilon)$, for all k, which means that Z has roughly the same slope at y_1^k as X at $k\varepsilon$ and that Z is weakly less convex than X at these points. This is crucial to the proof.

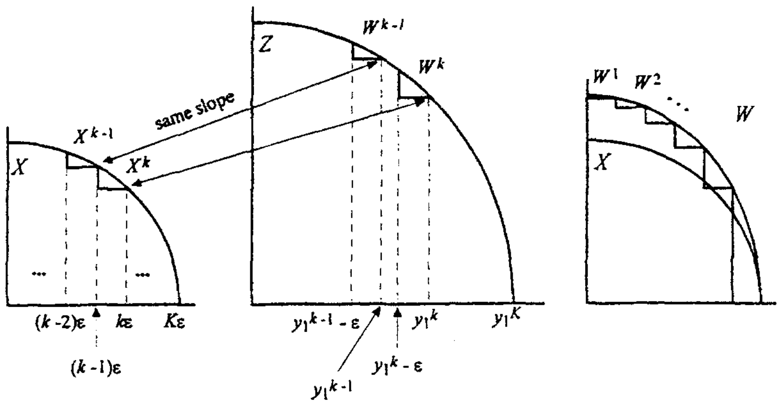

Fig. 5. Partitioning X and Z

Let $W^k \equiv [Z-(y_1^k-\varepsilon, bZ(y_1^k))]\cap \mathbf{R}^2_+$, for $k=1,2,\ldots,K-1$. Define $y_1^0 \equiv 0$ and let $V^k \equiv [Z-(y_1^{k-1}, bZ(y_1^k-\varepsilon))]\cap \mathbf{R}^2_+$, for $k=1,2,\ldots,K$. By construction, $\{\{V^k\}_{k=1}^K, \{W^k\}_{k=1}^K\}$ is a boundary partition of Z. Let $W \equiv \sum_{k=1}^K W^k$ and $V \equiv \sum_{k=1}^K V^k$. We have, then, that $(W, V) \in P(Z)$. (Figure 5 diagrams the construction of these sets.)

Step 3: Comparing W and X. Each W^k was constructed to make it "look like" X^k. These sets have a couple of noteworthy properties. First, $m_1 W^k = m_1 X^k = \varepsilon$, for all k. Second, $bW^k(x_1) - bX^k(x_1)$ is non-negative and non-increasing in x_1. (This follows from lemma 1; *eff*(W^k) is less concave than *eff*(X^k).) Notice that the W^k's and X^k's define the boundaries of W and X, respectively, in sequence. Therefore, $bW(x_1) - bX(x_1)$ is non-negative and non-increasing in x_1. Also, $m_1 W = m_1 X$. These facts imply that the Hausdorff distance between X and W is bounded above by $bW(0) - bX(0)$.

Take any $k \neq K$. Note that $bW^k(0) - bW^k(\varepsilon)$ is bounded above by $-\varepsilon d_- W^k(\varepsilon)$, which is no greater than $-\varepsilon d_+ X^k(\varepsilon)$, by the construction of W^k. (Remember that the slopes are non-positive.) Also, $bX^k(0) - bX^k(\varepsilon)$ is bounded below by $-\varepsilon d_+ X^k(0)$. Furthermore, $bW^k(\varepsilon) = bX^k(\varepsilon) = 0$. Therefore,

$$bW^k(0) - bX^k(0) = bW^k(0) - bW^k(\varepsilon) - [bZ^k(0) - bX^k(\varepsilon)] \leq \varepsilon[d_+X^k(0) - d_+X^k(\varepsilon)],$$

for $k = 1,2,\ldots,K-1$. It is also obvious that $bW^k(0) - bX^K(0) \leq bW^K(0) \leq -\varepsilon d_- Z(m_1 Z)$. Thus

$$bW(0) - bX(0) = \sum_{k=1}^{K} [bW^k(0) - bX^k(0)] \leq \sum_{k=1}^{K-1} \varepsilon[d_+X^k(0) - d_+X^k(\varepsilon)] - \varepsilon d_- Z(m_1 Z).$$

Note that $d_+X^k(0) = d_+X^{k-1}(\varepsilon)$ for each $k = 2, \ldots, K$, by the construction of each X^k. In addition, $d_+X^1(0) < 0$. Therefore,

$$bW(0) - bX(0) \leq -\varepsilon d_+X^{K-1}(\varepsilon) - \varepsilon d_-Z(m_1 Z).$$

Since $(K-1)\varepsilon < m_1 X$, it is the case that $d_+X^{K-1}(\varepsilon) \geq d_-X(m_1 X)$. Therefore,

$$bW(0) - bX(0) \leq -\varepsilon[d_-X(m_1 X) + d_-Z(m_1 Z)].$$

Step 4: Y' is defined as the limit of V as $K \to \infty$. Index W and V by ε to note the dependence on ε, and note that ε can be made arbitrarily small by letting K approach ∞. We have, then, that W_ε converges to X in the Hausdorff metric (as ε approaches zero). This implies that V_ε must converge to some $Y' \in \mathbf{B}$. Since $(W_\varepsilon, V_\varepsilon) \in P(Z)$, we have that $(X, Y') \in \bar{P}(Z)$. *Q.E.D.*

Proof of Lemma 4: We shall prove the theorem for strong 1-reductions. The proof for strong 2-reductions is analogous.

First we must show that if Z' is a 1-reduction of Z and $p_+Z'(x_2) \leq p_+Z(x_2)$ for all $x_2 \in [0, m_2 Z]$, then Z' is a strong 1-reduction of Z. Take any finite boundary partition of Z, $\{V^1, V^2, \ldots, V^n\}$. This partition is characterized by numbers $x_2^0, x_2^1, \ldots, x_2^n$, where $0 = x_2^0 < x_2^1 < \cdots < x_2^n = m_2 Z$. That is, V^k is constructed from the boundary of Z between x_2^{k-1} and x_2^k. (Thus, $m_2 V^k = x_2^k - x_2^{k-1}$.) Since $m_2 Z' = m_2 Z$, $\{x_2^0, x_2^1, \ldots, x_2^n\}$ also defines a boundary partition of Z', which we denote $\{W^1, W^2, \ldots, W^n\}$. We have that $m_2 W^k = m_2 V^k$ for each k. In addition, the slope condition guarantees that $W^k \subset V^k$ for each k, so W^k is a 1-reduction of V^k. Now partition $\{1, 2, \ldots, n\}$ into two sets, K' and K'', and let $X \equiv \sum_{k \in K'} V^k$, $Y \equiv \sum_{k \in K''} V^k$, $X' \equiv \sum_{k \in K'} W^k$, and $Y' \equiv \sum_{k \in K''} W^k$. We have that $X + Y = Z$ and $X' + Y' = Z'$. Furthermore, $m_2 X = m_2 X'$, $m_2 Y = m_2 Y'$, $X' \subset X$, and $Y' \subset Y$. Therefore, X' is a 1-reduction of X and Y' is a 1-reduction of Y.

This construction implies that for any sets $X, Y \in \mathbf{B}'$ that are formed from a finite boundary partition of Z, we can find sets X', $Y' \in \mathbf{B}'$ such that X' is a 1-reduction of X, Y' is a 1-reduction of Y, and $X' + Y' = Z'$. Now take any $X, Y \in \mathbf{B}'$ such that $X + Y = Z$. By lemma 3 we know that $(X, Y) \in \bar{P}(Z)$, so we can find a sequence $\{X^{(k)}, Y^{(k)}\}_{k=1}^{\infty}$, where each $(X^{(k)}, Y^{(k)})$ is formed from a finite boundary parition of Z, such that $(X^{(k)}, Y^{(k)})$ converges to (X, Y). The above construction implies the existence of a sequence $\{X'^{(k)}, Y'^{(k)}\}$ of sets formed from finite boundary partitions of Z', for which $X'^{(k)}$ is a 1-reduction of $X^{(k)}$ and $Y'^{(k)}$ is a 1-reduction of $Y^{(k)}$, for each k. Furthermore, since Z' is bounded and each $(X'^{(k)}, Y'^{(k)})$ is convex and closed, we can find a subsequence $\{X'^{(k_l)}, Y'^{(k_l)}\}_{l=1}^{\infty}$ that converges in the Hausdorff metric. We thus have that $X^{(k_l)} \to X$, $Y^{(k_l)} \to Y$, $X'^{(k_l)} \to X'$, and $Y'^{(k_l)} \to Y'$, for some X' $Y' \in \mathbf{B}'$ such that $X' + Y' = Z'$. Since $X'^{(k_l)}$ is a 1-reduction of $X^{(k_l)}$ and $Y'^{(k_l)}$ is a 1-reduction of $Y^{(k_l)}$ for each k, it follows that X' is a 1-reduction of X and Y' is a 1-reduction of Y. This proves the first half of the lemma.

To prove that second half of the lemma, we show that if Z' is a 1-reduction of Z yet $p_+Z'(x_2) > p_+Z(x_2)$ for some $x_2 \in [0, m_2 Z]$, then Z' cannot be a strong

1-reduction of Z. Take such a $Z'\in\mathbf{B}'$ and let z_2 be such that $p_+Z'(z_2) > p_+Z(z_2)$. We can then find a number $z_2' < z_2$ for which $p_-Z'(z_2') > p_+Z(z_2)$. Let z_1' be such that $z' = (z_1', z_2')\in eff(Z')$ and let z_1 be such that $z = (z_1, z_2)\in eff(Z)$. Next define $X \equiv [Z - (g_1(z, Z), z_2)] \cap \mathbf{R}^2_+$ and $Y \equiv [Z - (z_1, g_2(z, Z))] \cap \mathbf{R}^2_+$. We have that $X + Y = Z$.

Presume that Z' is a strong 1-reduction of Z and we will find a contradiction. Let X' and Y' be such that $X' + Y' = Z'$, X' is a 1-reduction of X, and Y' is a 1-reduction of Y. Note that $m_2 Y = m_2 Y' = z_2$. Let $x'\in eff(X')$ and $y'\in eff(Y')$ be such that $x' + y' = z'$. Lemma 1 implies that $p_- Y'(y_2') \geq p_- Z'(z_2')$ and we know that $p_- Z'(z_2') > p_+ Z(z_2) = p_+ Y(z_2)$. Since $x_2' + y_2' < z_2$ and $x_2' \geq 0$, it must be that $y_2' < z_2$. Because Y' is convex, this implies that $p_+ Y'(z_2) \geq p_- Y'(y_2')$. Therefore, $p_+ Y'(z_2) > p_+ Y(z_2)$. Since $m_2 Y = m_2 Y' = z_2$, this contradicts that $Y' \subset Y$, which proves the second half of the result. *Q.E.D.*

B Proof of Theorem 3

First, take any solution f that satisfies EFF, SYM, and IIAI; we will show that it must be symmetric utilitarian. Begin by taking any set $X\in\mathbf{B}$ such that $(0, m_2 X)$ is the unique point in X that maximizes the sum of the players' utilities (over X). We need to demonstrate that $(0, m_2 X) = f(X)$. Let $X^c \equiv \{(x_2, x_1) | (x_1, x_2)\in X\}$. Notice that $d_- X^c(m_1 X^c) \geq d_+ X(0)$. By construction, $X + X^c$ is a symmetric set, so SYM and EFF imply that $f(X + X^c) = (m_2 X, m_2 X)$. Furthermore, $X + X^c$ is strictly convex at the point $(m_2 X, m_2 X)\in eff(X + X^c)$, which implies that for $x\in X$ and $x'\in X^c$, $x + x' = (m_2 X, m_2 X)$ only if $x = (0, m_2 X)$ and $x' = (m_2 X, 0)$. Also, for $x\in X$, $comp(x + X^c) \cap eff(X + X^c) = \emptyset$ unless $x = (0, m_2 X)$. By IIAI, $f(X + X^c) = f(comp(f(X) + X^c))$, and by EFF, $f(X + X^c)\in eff(X + X^c)$. Thus, it must be that $f(X) = (0, m_2 X)$. An analogous argument shows that $f(Y) = (m_1 Y, 0)$ for any Y for which $(m_1 Y, 0)$ uniquely maximizes the sum of the players' utilities.

Next take any set Z for which there is some $z\in Z$ such that $z \gg 0$ and z uniquely maximizes the sum of the players' utilities (over Z). We must show that $f(Z) = z$. Let $X \equiv [Z - (z_1, 0)] \cap \mathbf{R}^2_+$ and let $Y \equiv [Z - (0, z_2)] \cap \mathbf{R}^2_+$. By construction, using the analysis above, $f(X) = (0, z_2)$ and $f(Y) = (z_1, 0)$. The EFF and IIAI axioms imply that

$$f(Z) = f(X + Y) = f(comp((0, z_2) + Y))\in eff[comp((0, z_2) + Y)]$$

and

$$f(Z) = f(X + Y) = f(comp((z_1, 0) + X))\in eff[comp((z_2, 0) + X)].$$

Also, $f(X + Y)\in eff(X + Y)$. Note that

$$eff[comp((0, z_2) + Y)] \cap eff[comp((z_1, 0) + X)] \cap eff(X + Y) = \{z\},$$

which implies that $f(Z) = z$.

Finally, take any set $W \in \mathbf{B}$ for which there is not a unique member that maximizes the sum of the players' utilities. In this case, there will be a linear, closed interval of points in $eff(W)$ that maximizes the sum of the players' utilities over W. Let w and w' be the extreme points of this interval, defined so that $w_1 > w'_1$. We must prove that $f(W) = ((w_1 + w'_1)/2, (w_2 + w'_2)/2) \equiv w$. Let $X \equiv [W - (w_1, 0)] \cap \mathbf{R}^2_+$, let $Y \equiv [W - (0, w'_2)] \cap \mathbf{R}^2_+$, $S \equiv [W - (w'_1, w_2)] \cap \mathbf{R}^2_+$, and $Z \equiv X + Y$.

We have constructed these sets so that $W = Z + S$. It must be that $w_1 - w'_1 = w'_2 - w_2$, and so S is symmetric with $f(S) = ((w_1 - w'_1)/2, (w'_2 - w_2)/2) \equiv s$. Furthermore, by construction, $(w'_1, w_2) \equiv z$ uniquely maximizes the sum of the players' utilities over points in Z, and so Z is covered by the analysis of the last paragraph. Therefore, $f(Z) = z$. We have, by IIAI, that

$$f(comp(z + S)) = f(W) = f(comp(s + Z)).$$

But by EFF,

$$f(comp(z + S)) \in eff[comp(z + S)]$$

and

$$f(comp(s + Z)) \in eff[comp(s + Z)].$$

It is easy to see that $eff[comp(z + Z)] \cap eff[comp(s + Z)] = \{s + z\} = \{w\}$. Therefore, $f(W) = w$.

It is not difficult to check that the symmetric utilitarian solution satisfies the EFF, SYM, and IIAI axioms, which completes the proof. *Q.E.D.*

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