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Search and Bargaining in Large Markets With Homogeneous Traders

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Abstract

We study decentralized trade in dynamic markets with homogeneous, non-atomic, buyers and sellers that wish to exchange one unit. In the first part of the paper we characterize equilibrium in a bargaining model with two-sided time varying outside options. In the second part we analyze a market equilibrium model in which (i) buyers and sellers are randomly matched in pairs; (ii) each buyer-seller pair bargains over the price of a good; and (iii) each agent has the option of abandoning negotiations, in which case the value of returning to the pool of unmatched agents constitutes an outside option. The second part is therefore an application of the first part in which the values of the outside options are endogenous to the model. Conditions for uniqueness of the market equilibrium are given; when it is unique it converges to the Walrasian outcome as frictions vanish. To the extent that multiplicity of market equilibria may (under some conditions) persist as frictions are removed, the limit of some sequences of equilibrium prices may converge to non-Walrasian values.

KEYWORDS: Bargaining, Markets, Outside Options

1 Introduction

This paper is a contribution to the theory of price formation in markets with decentralized trade and bargaining, focusing on the distinctive implications of voluntary search. We consider an economy à la Rubinstein and Wolinsky [1985] where homogeneous buyers and sellers that wish to exchange one unit are randomly matched in pairs, and each pair bargains over the price at which to trade. Each period, agents that trade leave the market, agents that fail to trade stay and new agents enter. Buyer-seller pairs play an alternating offer bargaining game. Upon rejection of an offer, both players may quit the match. If one player quits a match, then both she and her partner return to the pool of traders that will be randomly matched in the following period. If neither decides to search for an alternative partner, they remain matched and continue bargaining. Thus, bargaining is an infinite horizon game with two-sided outside options, the market options, whose value over time is determined by the strategies of traders throughout the market. Markets that remain stationary as the flow of entry exactly matches the measure of consummated trades are only a particular case; generally non-stationary market equilibria must be considered. When a market is non-stationary, the value of the market options varies over time.

Bargaining games with two-sided outside options that change over time have not been studied in the literature[1]. Consequently, the first step of our analysis is to characterize subgame perfect equilibria in such bargaining games. We provide such a characterization and subsequently build on it as we focus our attention on market equilibria. Equilibria for games with two-sided outside options of constant value are characterized in Ponsatí and Sákovics [1998]. Their fundamental insight is that (although the bargaining protocol permits an infinite sequence of alternating proposals) the set of subgame perfect equilibria always contains a profile of ultimatum strategies, namely one in which the proposer offers the responder a share of value equal to the outside option, and she accepts.[2] We show that the main arguments of Ponsatí and Sákovics [1998] carry over when outside options vary over

[1]Models of bargaining in non-stationary environments, as Merlo and Wilson [1995], and Coles and Muthoo [2003], do not explicitly consider the possibility that bargainers opt out.

[2]The ultimatum profile is generally not the unique subgame perfect equilibrium, though; other subgame perfect equilibria can prevail when the combined values of the anticipated outside options of both bargainers are not too great.

time. Hence the existence of an equilibrium in ultimatum strategies can still be assured, allowing an analogous characterization of the equilibrium for the more general environment.

The observation that bargaining games admit an equilibrium in ultimatum strategies has important implications as we consider market equilibria: The use of ultimatum strategies by all traders in a market results in a unique sequence of market options. Under this endogenous sequence of market options, playing ultimatum strategies is a subgame perfect equilibrium for all bargaining pairs. Therefore a market equilibrium in ultimatum strategies always exists. For an important subset of environments it is the unique market equilibrium: In fact, in spite of the (potentially) large set of subgame perfect equilibria of the bargaining game, the endogenous nature of the market options drastically limits the range of market equilibria. In stationary markets, for instance, the market equilibrium in ultimatum strategies prevails uniquely when search and bargaining costs are not too high (and coincides with the unique equilibrium in Rubinstein and Wolinsky [1985] as long as search costs are independent of bargaining costs). In non-stationary environments, if search costs are low and players are impatient, then the market equilibrium in ultimatum strategies prevails as the unique market equilibrium as well. In general, however, if players are not too impatient a continuum of market equilibria exists and a non-degenerate interval of equilibrium prices can be supported.

In non-stationary markets, as frictions vanish, prices along the market equilibrium in ultimatum strategies approach the Walrasian price, an asymptotic prediction consistent with the results of Binmore and Herrero [1988] and Gale [1987]. Nevertheless, multiple equilibria can persist in the limit provided that frictions vanish along a path where search costs stay higher than bargaining costs. Consequently, there exist sequences of market equilibria where, as frictions vanish, prices fail to converge to the Walrasian price.

The non-cooperative foundations of Walrasian and/or efficient outcomes in markets where trade takes place in decentralized pair-wise meetings is a fundamental issue that has been the object of important contributions and heated controversy.[3] Our model is most closely related to Binmore and Herrero [1988] that addresses, as we do, non-steady states for markets with non-atomic and homogeneous agents. The main difference is their

[3] See Osborne and Rubinstein [1990] for an excellent survey.

assumption that all active agents participate in the matching process each period. Since search is not voluntary in their model, their bargaining pairs effectively play an ultimatum game, and thus their unique equilibrium is closely related to our market equilibrium in ultimatum strategies. Search is voluntary in the models of Gale [1987] ,Wolinsky [1987] , Bester [1988] , and Muthoo [1993], among others. While these contributions discuss a large variety of environments, they share two main features. First, the message that, as frictions disappear, market equilibria converge to the Walrasian outcome. And second, the assumption that players cannot break up a bargaining match in periods where they act as the proposer. As weak as such a constraint might appear, it is not.[4] Shaked [1994] sharply makes this point in a model where a unique seller confronts many buyers; only the seller can quit bargaining and she can do it while being the proposer.[5] Shaked's model yields a continuum of market equilibria (provided that the search costs of the seller are neither too small nor too large); multiple equilibria (some with prices well above the Walrasian price) remain as frictions vanish.

Our model bridges the gap between Shaked and the rest of the literature since we consider markets where proposers can exit without delay, but we rule out market power (there is a continuum of buyers and a continuum of sellers) and strategic asymmetries (our bargaining games treat traders symmetrically). Away from the drastic asymmetries of Shaked's set up, we can establish natural conditions assuring that the Walrasian outcome prevails uniquely as frictions vanish. However, we can also display natural environments where the right to quit of proposers remains the source of multiple market equilibria, even in the limit.

In as much as frictionless markets are identified with the Walrasian outcome, we must conclude that this identification is tantamount to the assertion that agents play ultimatum strategies. While our findings support ultimatum games as a reduced form of more complex bargaining games, they also point out that this simplifying approach overlooks interesting (and asymptotically persistent) equilibria.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes subgame perfect equilibria of alternating offer bargaining games with two-sided time-varying outside options. Section

[4]That minor procedural changes have great impact is well known. See, for example, Muthoo [1999] , chapter 7.

[5]See also the discussion of Shaked's model in Osborne and Rubinstein [1990] , section 3.12.2.

4 characterizes market equilibria in non-stationary environments. Steady states are characterized in Section 5. Section 6 concludes.

2 The model

Consider a market where trade is carried out by decentralized agreements of buyer-seller pairs that meet randomly over time and negotiate the price to trade one unit of an indivisible good. Time is measured in discrete equally spaced periods $t = 0, 1, 2, ...$ and players discount the future with a common discount parameter δ, $0 < \delta < 1$. If a pair trades the good at price p in period t, the seller obtains payoff $p\delta^t$ and the buyer obtains $(1-p)\,\delta^t$.

There is a continuum of buyers and sellers. That is, there are initial measures of buyers and sellers b_0 and s_0 and a constant flow of entry of buyers and sellers, $z = (b, s)\,, b \geq 0$ and $s \geq 0$. Formally, the set of buyers in the market, denoted by I_b, is

$$I_b = [0, b_0] \cup \left(\cup_{k=0}^{\infty} [b_0 + kb, b_0 + (k+1)b] \right)$$

and the set of sellers is

$$I_s = [0, s_0] \cup \left(\cup_{k=0}^{\infty} [s_0 + ks, s_0 + (k+1)s] \right).$$

The state of the market at t, $z_t = (b_t,\, s_t)$, is the pair of measures of buyers and sellers searching for a partner at the beginning of each date t. Without loss of generality, we assume that sellers are initially the short side of the market, $b_0 = 1$ and $s_0 \leq 1$, and that $b \geq s \geq 0$.[6] Given the initial populations, as traders leave the market in buyer-seller pairs, sellers remain the short side of the market. Thus as $t \to \infty$, unless $s = b$ and this measure coincides with the measure of realized trades (i.e. as in Rubinstein and Wolinsky [1985]), the market cannot remain stationary and $\frac{s_t}{b_t} \to 0$.

At each period $t \geq 0$, unmatched sellers and buyers meet at random. The probabilities of finding a trading counterpart at t are denoted π_{bt} and π_{st}, $0 < \pi_{bt} < \pi_{st} < 1$, respectively for buyers and sellers. For the sake

[6] Symmetric results hold in the opposite case. Since entry flows are stationary, assuming that the initial short side is also the short side in the entry process is almost without loss of generality. If entry were higher in the initial short side, it would remain the short side only for finitely many periods.

of simplicity, when discussing non-stationary environments we assume that sellers find a match with a constant probability, $\pi_{st} = \gamma, 0 < \gamma < 1$, and that buyers meet a seller with probability $\pi_{bt} = \gamma \frac{s_t}{b_t}$.[7] Search frictions are said to vanish if the probability that sellers find a match approaches one, $\gamma \to 1$.

When a buyer and a seller meet, they play an alternating proposals bargaining game to set a price for the good. Each is selected to act as first proposer with probability $\frac{1}{2}$. Acceptance of a proposal ends the game, agents trade and leave the market. Upon rejection, the bargaining pair may continue bargaining if both wish to do so; otherwise, either may break the match and then both return to the pool of unmatched agents.[8] Thus, the bargaining game has two-sided outside options which are (potentially) time varying and endogenously determined as the present value of reentering the pool of unmatched agents.

Since there is a continuum of buyers and sellers, agents that quit bargaining pairs prior to trade find their previous partners with probability 0, so that, without loss of generality, we can assume that each bargaining pair plays a game that is independent of the history of the agents in previous meetings. A *bargaining strategy* specifies actions (a proposed price/a decision to accept or reject/a decision to stay or leave the match) for each subgame. Consequently the value of the outside options to a player in a given bargaining pair is unaffected by her own (or the opponent's) bargaining strategy *in that bargaining pair*. Thus, from the point of view of each bargaining pair, market outside options can be taken as exogenous. *Subgame perfect equilibria* of a bargaining game are pairs of bargaining strategies that are best response to each other at each subgame.

A *buyers' market profile of strategies* is a (measurable) function specifying a bargaining strategy for each buyer

$$\beta : I_b \to \Sigma$$

where Σ denotes the set of bargaining strategies.[9] A *sellers' market profile*

[7] Our results are robust to more general formulations. A reasonable alternative is to assume that π_{st} decreases in $\frac{s_t}{b_t}$, that is, as sellers become (relatively) increasingly scarce and buyers increasingly abundant, sellers find a match more easily.

[8] The decisions to break a match after a proposal is rejected may be sequential or simultaneous. See discussion in section 3.

[9] Assuming that Σ is common to all players is without loss of generality because the

of strategies, denoted σ, is defined similarly, as

$$\sigma : I_s \to \Sigma.$$

A pair (β, σ) is a *market profile of strategies*. Note that the previous definitions implicitly assume that the same bargaining strategy is used in all bargaining games that a trader might play, and that bargaining strategies are not contingent on the opponent's identity. Given a market profile (β, σ) the measure of agents that trade at t and the (distribution of) prices at which these trades occur are uniquely determined - depending only on the bargaining strategies played by subsets of active traders that have positive measure. For each buyer i (seller j), the *market option at* t given the market profile (β, σ), denoted $x^i_{bt}(\beta, \sigma)$ (respectively $x^j_{st}(\beta, \sigma)$), is the expected value of opting out from a bargaining pair at t when all agents play according to the market profile of strategies thereafter. A market profile (β, σ) is a *market equilibrium* if and only if, for all t, and for almost all pairs (i, j) that are active in the market at t along (β, σ), the strategy pair $(\beta(i), \sigma(j))$ is a subgame perfect equilibrium of the bargaining game with outside options $\left\{x^i_{bt+k}(\beta, \sigma), x^j_{st+k}(\beta, \sigma)\right\}_{k=0}^{\infty}$.

Two special cases are important: at a *uniform market profile* (almost) all buyers use the same bargaining strategy and so do all sellers; at an *immediate agreement profile* (almost) all bargaining pairs reach an immediate agreement. At a uniform market profile all buyers/sellers that initiate bargaining at the same date τ have the same market option at t, for all $t \geq \tau$. At a *uniform immediate agreement profile* the outside options are identity independent; that is, $x^i_{bt}(\beta, \sigma) = x_{bt}(\beta, \sigma)$ for all $i \in I_b$ and $x^j_{st}(\beta, \sigma) = x_{st}(\beta, \sigma)$ for all $j \in I_s$.

When a uniform immediate agreement profile is a market equilibrium we will say that the associate market options $\{x_{bt}(\beta, \sigma), x_{st}(\beta, \sigma)\}_{t=0}^{\infty}$ are the *market equilibrium options*. Consequently, an arbitrary sequence $\{x_{bt}, x_{st}\}_{t=0}^{\infty}$ is a sequence of market equilibrium options if there is a uniform immediate agreement profile (β, σ) such that $x_{bt} = x_{bt}(\beta, \sigma)$ and $x_{st} = x_{st}(\beta, \sigma)$ for all t, and (β, σ) is a market equilibrium.

Addressing situations in which buyer-seller pairs play an alternating offers bargaining game with endogenous and time varying outside options

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actions available at each subgame remain constant through bargaining games that differ in their initial date and outside options. Of course, for an agent $i \in [1 + kb, 1 + (k+1)b]$, i.e. one that enters the market at $t = k$, a bargaining strategy specifies actions only at subgames with dates $t' \geq k$.

demands a characterization of subgame perfect equilibrium outcomes with exogenous time varying outside options. A detailed discussion of bargaining games with exogenous, time invariant, and certain options that can be taken at all times by both players is in Ponsatí and Sákovics [1998][10]. In what follows we consider the more general case of two-sided outside options that are given by infinite sequences of known values.

3 Bargaining with two-sided, time-varying, outside options

Consider an alternating proposal bargaining game à la Rubinstein, where upon any rejection both players can opt out and terminate the game. Payoffs upon agreement are as usual. When one player opts out, then both players receive the outside option. The values of the outside options are given by a pair of infinite sequences $\{x_{1t}, x_{2t}\}$, where x_{1t} denotes the outside option at t of the player chosen to be the first proposer (respectively, x_{2t} denotes the option of the first responder). Specifying the timing (sequential or simultaneous) of exit decisions completes the description of the bargaining game with two-sided and time varying outside options.

We assume that the sequences of outside options satisfy $x_{it} \geq \delta x_{it+1}$ for both players and all t. This is a most natural assumption: a player always has the "option at t" of waiting for the option that comes at $t+1$, and this option has value δx_{it+1} at t, thus δx_{it+1} must be a lower bound of the outside options of player i at t.

Observe that if $x_{1t} + x_{2t} > 1$ at some t, equilibrium requires that players take their options at that t. Then the game turns effectively into one with a finite horizon, and the (unique) subgame perfect equilibrium strategies are obtained by a straightforward backward induction. Consequently, in what follows we restrict attention to the case that $x_{1t} + x_{2t} \leq 1$ for all t.

When the strategy of agent i is such that she always proposes $(1 - x_{jt}, x_{jt})$ and takes the outside option upon any rejection of her opponent, we will say that she plays an *ultimatum strategy*. An agent that uses an ultimatum strategy chooses proposals as if she was in an ultimatum game, where any rejection automatically terminates the game. The key observation that

[10] The case of uncertain outside options is discussed in Ponsatí and Sákovics [1999] and Ponsatí and Sákovics [2001] .

drives our results is the following simple but powerful lemma, that applies independently of the specific timing of exit decisions. It says that an equilibrium in ultimatum strategies always exists.

Lemma 1 *If player* 1 *proposes at* t*, immediate agreement on* $(1 - x_{2t}, x_{2t})$ *is an outcome that can be supported by a subgame perfect equilibrium.*

Proof: The following ultimatum strategy profile is a subgame perfect equilibrium. If Player i is the proposer he always asks for $1-x_{jt}$; the responder accepts any proposal that is not worse than the (candidate) equilibrium proposal; if the proposer asks for more, then the responder rejects and takes her outside option. Upon any rejection the proposer opts out.

Note first that in anticipation of a period $t+1$ offer that pays δx_{it+1} the proposers' decision to opt out at t upon rejection of any proposal is optimal since $x_{it} \geq \delta x_{it+1}$; and observe that this holds regardless of the specific sequence of exit decisions. Since the proposer's decision to opt out leaves the responder at her outside option as well, the acceptance of a proposal that pays the value of the outside option is optimal; while the responder's decision about exit is irrelevant. It is then optimal that the proposer demands as much as $1 - x_{jt}$. Consequently, immediate agreement at $(1 - x_{2t}, x_{2t})$ is an outcome that can be supported by a subgame perfect equilibrium for all specifications of the game. ■

The driving force behind Lemma 1 is the ability of the proposer to opt out in the same time period that her proposal is rejected, combined with the alternation in proposer-responder roles. Proposers can make a demand and threaten to leave if it is rejected. The credibility of such threat depends on the proposer's expected value if she continues bargaining next period, and not on the value of the rejected demand. When the proposer's threat can be sustained, the responder must accept any proposal that pays at least as much as her outside option. If outside options were taken by the decision of one player only, her threat to opt out would not be credible unless her outside option was sufficiently valuable - because staying after a rejection she would obtain a reasonable payoff even as a responder. But in our formulation the current responder is next period's proposer (and has the ability to threaten in the continuation). Hence in an ultimatun profile the current proposer expects little if bargaining continues and the most extreme demand can be sustained with a credible threat. In summary, the ability of both players to opt out without delay when their proposal is rejected grants credibility to mutual threats to do so, and eliminates any bargaining power of the responders beyond the outside option.

What other strategy profiles be sustained as a subgame perfect equilibrium? The outside options are a lower bound to the payoffs that agents attain in subgame perfect equilibria. Consequently, if the outside options are sufficiently valuable, Lemma 1 describes the unique subgame perfect equilibrium outcome, because the unique best response of the proposer upon the hypothetical rejection of any proposal is to opt out.[11] When the options have low value, however, other subgame perfect equilibria - in which players do not use the ultimatum strategies of Lemma 1 - are possible. Along such equilibria responders obtain payoffs strictly above their (low) outside option; therefore a proposer that sees her proposal rejected expects more than the outside option value if she stays in the negotiation for another period. As proposers cannot credibly threaten to opt out, responders obtain the continuation value of staying in the negotiation for another period, which indeed is strictly greater than the outside option value. Therefore, under outside options of low value, there is a continuum of equilibria.

The range of payoffs that a player can attain in equilibrium must surely be contained in the interval between her outside option value and her ultimatum share. But not all the values of this interval can be sustained in a subgame perfect equilibrium. The precise range is given in Proposition 2 that follows.

Proposition 2 *Consider (sub)games starting at t where Player i is the proposer. i) If $x_{it} + \delta x_{jt+1} \leq \delta^2$ player i can obtain a payoff s in a subgame perfect equilibrium if and only if $s \in [(1-\delta) + x_{it}, 1 - x_{jt}]$. ii) If $x_{jt} + \delta x_{it+1} > \delta^2$ in the unique subgame perfect equilibrium outcome player i offers x_{jt} to j and she accepts.*

Proof: If there are equilibria with outcomes different from that of Lemma 1, they must be such that players remain in the match upon rejection of some proposals.

Fix a subgame perfect equilibrium and assume that there is a subgame at which, when the proposal of player i is rejected, i stays at $t+1$. Let v_{jt} denote an equilibrium payoff of the responder at t. Our first observation is that $v_{jt} \leq \delta \min\left\{(1 - x_{it+1}), (1 - \frac{x_{it}}{\delta})\right\}$ since j cannot expect more than the

[11] Provided that staying implies the continuation of bargaining after rejection, of course. If the responder opts out at rejection, staying is weakly a best response.

discounted value of the maximum payoff that she can obtain while acting as the proposer in the following period: $(1 - x_{it+1})$ is an upper bound to the shares that j can expect from i in period $t+1$; but this upper bound is not attained if x_{it+1} is below $\frac{x_{it}}{\delta}$ - because otherwise i would surely take the outside option upon any rejection (by j) at t. Now, $x_{it} \geq \delta x_{it+1}$ by assumption, so that $\min\left\{(1 - x_{it+1}), (1 - \frac{x_{it}}{\delta})\right\} = (1 - \frac{x_{it}}{\delta})$. Hence, it is necessary that $v_{jt} \leq \delta(1 - \frac{x_{it}}{\delta}) = \delta - x_{it}$. Consider now player i's exit decision upon rejection at t: if she stays (to be a responder at $t+1$) she can expect no more than $\delta(\delta - x_{jt+1})$, therefore if $x_{it} > \delta(\delta - x_{jt+1})$, i.e. $x_{it} + \delta x_{jt+1} > \delta^2$, then i must necessarily take the outside option at t upon any rejection; contradicting the alleged equilibrium behavior. Hence the existence of equilibrium outcomes different from those of Lemma 1 requires that $x_{it} + \delta x_{jt+1} \leq \delta^2$. This proves ii).

We now check that, when the game starts at t and i acts as the proposer, if $x_{it} + \delta x_{jt+1} \leq \delta^2$ then for all $s \in [(1-\delta) + x_{it}, 1 - x_{jt}]$ there is a subgame perfect equilibrium that yields a share s to i. Payoffs outside this interval cannot be sustained. The upper bound is immediate since j can opt out and get x_{jt}. The lower bound follows by the arguments of the preceding paragraph since $s \geq 1 - v_{jt}$ and $v_{jt} \leq \delta(1 - x_{it}/\delta)$ imply that $s \geq 1 - \delta(1 - x_{it}/\delta) = (1-\delta) + x_{it}$.

To complete the proof it suffices to check that the following strategies are a subgame perfect equilibrium, for each $s \in [(1-\delta) + x_{it}, 1 - x_{jt}]$. As proposer, player i offers a share $(1-s)$ to player j. In its turn, player j accepts any share greater or equal to $(1-s)$. If player j rejects a proposal, j stays in the game, and i opts out if and only if the rejected proposal offered j a share greater or equal to $(1-s)$. In the continuation that follows j's rejection a subgame perfect equilibrium sustaining the extreme outcome of Lemma 1 is played. ■

Remark 1 *For each equilibrium payoff we have exhibited a strategy profile that sustains it in which agreement is immediate. Equilibrium profiles where agreement is delayed can also be constructed.*[12]

In summary, the outside option principle[13] - that outside options of small

[12] The existence of equilibria with delay is a standard feature of bargaining games admitting an interval of equilibrium payoffs. As this is not our main interest we skip further discussion.

[13] See Shaked and Sutton [1984].

value are irrelevant so that the (unique) outcome of the game without options prevails - does not apply in the present environments: Although the bargaining game is not an ultimatum game a subgame perfect equilibrium in ultimatum strategies always exists. We now turn attention to the implications that equilibrium behavior within bargaining pairs has over market equilibria.

4 Market Equilibria

Consider a market profile where all traders use ultimatum strategies: as proposers, agents offer a share equal to their partners' market option, as responders they accept any offer of a share at least as great as their own market option, and in case of rejection all opt out. At this profile all matched pairs trade and the market options x_{st} and x_{bt} must satisfy[14] $x_{bt} = \delta\pi_{t+1}v_{bt+1} + \delta\left(1-\pi_{t+1}\right)x_{bt+1}$ and $x_{st} = \delta\gamma v_{st+1} + \delta\left(1-\gamma\right)x_{st+1}$, where v_{bt} and v_{st} denote the expected values of being in a bargaining pair at t (before the proposer has been selected), respectively for buyers and sellers; and where π_{t+1} are uniquely determined because all pairs trade immediately. Under ultimatum strategies, since buyers and sellers are chosen to be the initial proposer with equal probability, these expected values must solve $v_{bt} = \frac{1}{2}(1-x_{st}) + \frac{1}{2}x_{bt}$ and $v_{st} = \frac{1}{2}(1-x_{bt}) + \frac{1}{2}x_{st}$. Consequently, the market options must solve the following system of difference equations:

$$\begin{aligned} x_{bt} &= \delta\left(\tfrac{\pi_{t+1}}{2}(1-x_{st+1}) + (1-\tfrac{\pi_{t+1}}{2})x_{bt+1}\right) \\ x_{st} &= \delta\left(\tfrac{\gamma}{2}\left(1-x_{bt+1}\right) + \left(1-\tfrac{\gamma}{2}\right)x_{st+1}\right). \end{aligned} \tag{1}$$

The following is proved in the Appendix:

Lemma 3 *There is a unique sequence $\{x_{bt}, x_{st}\}$ solving (1). This unique solution can be written*

$$\begin{aligned} x_{st} &= \tfrac{\gamma}{2}\sum_{\tau=0}^{\infty}\delta^{\tau+1}(1-a_{t+1+\tau}) \\ x_{bt} &= \tfrac{1}{2}\sum_{\tau=0}^{\infty}\delta^{\tau+1}\pi_{t+1+\tau}(1-a_{t+1+\tau}) \end{aligned} \tag{2}$$

where $a_t = x_{bt} + x_{st}$ uniquely solves

$$a_t = \delta\left((1-\Delta_{t+1}) + \Delta_{t+1}a_{t+1}\right) \tag{3}$$

[14] The subindex b in π_{bt} is omitted for simplicity.

with

$$\Delta_t = 1 - \frac{\pi_t + \gamma}{2}. \quad (4)$$

Remark 2 *The outside options at* t*,* x_{st} *and* x_{bt}*, are simply the present value of the surplus exceeding the market options that each agent expects to appropriate from* $t+1$ *onwards.*

Remark 3 *As* $t \to \infty$*,* $a_t \to \frac{\delta\gamma}{2(1-\delta)+\delta\gamma}$*,* $x_{st} \to \frac{\delta\gamma}{2-\delta\gamma}$ *and* $x_{bt} \to \frac{\delta 2(1-\gamma)}{2-\delta\gamma}$*.*

With the value of the market options at hand, we are ready to establish that the uniform profile where all traders use ultimatum strategies is a market equilibrium:

Proposition 4 *A unique market equilibrium in ultimatum strategies exists.*

Proof: Assume that all agents in the market play ultimatum strategies. Then the market options are uniquely given as in Lemma 3. Observe that since $a_t < 1$ (which is shown to hold in the proof of Lemma 3), equation (1) implies that $x_{st} \geq \delta x_{st+1}$ and $x_{bt} \geq \delta x_{bt+1}$. Thus, by Lemma 1, the pair of ultimatum strategies is a subgame perfect equilibrium of the bargaining game for each match and thus the market profile constitutes a market equilibrium. ■

Are there other market equilibria? By Proposition 2, when $x_{it} > \delta(\delta - x_{jt+1})$ $i, j = 1, 2$ for all t, each bargaining pair must play ultimatum strategies. Hence, when the market options given in Lemma 3 satisfy the preceding condition, the market equilibrium in ultimatum strategies is the unique market equilibrium. That is, however, a very stringent condition. If it is not satisfied, other market equilibria - in which some traders use strategies that yield market options other than those of Lemma 3 - may exist. As long as the bargaining games with two-sided outside options admit a continuum of subgame perfect equilibria, it is possible that - in addition to the market equilibrium in ultimatum strategies - other market profiles can be sustained as market equilibria. To address this question we set upper and lower bounds to the payoffs that traders can attain in a market equilibrium.

At a market equilibrium, for each buyer-seller pair the bargaining game starting in period t yields expected subgame perfect equilibrium values v_{st} and v_{bt}, in $[\underline{v}_{st}, \overline{v}_{st}]$ and $[\underline{v}_{bt}, \overline{v}_{bt}]$, where $\overline{v}_{st}$ and $\overline{v}_{bt}$ ($\underline{v}_{st}$ and $\underline{v}_{bt}$) denote the supremum (infimum) of the payoffs that sellers and buyers can obtain at bargaining games starting at t in any market equilibrium.

Observe that in order to evaluate the extreme values that traders can attain at each t in a market equilibrium there is no loss of generality if we restrict attention to uniform immediate agreement market profiles. The sequence $\{\pi_t\}$ is the same at all such market profiles, and it is uniquely determined by γ, the initial measures of buyers and sellers and the flows of entry. At a uniform profile, at each t, all buyers (and similarly all sellers) obtain the same market option if they abandon a bargaining match. These market options must lie in intervals $[\underline{x}_{st}, \overline{x}_{st}]$ and $[\underline{x}_{bt}, \overline{x}_{bt}]$.

Note that $\overline{x}_{st} \leq \delta\gamma\overline{v}_{st+1} + \delta(1-\gamma)\overline{x}_{st+1}$ since the best market option cannot exceed the continuation payoffs obtained when, in the immediate posterior period, the best equilibrium outcome prevails in bargaining whenever a match occurs and the best continuation market option is obtained in the absence of a match. Moreover, since options are endogenously given and agents are always free to reenter the pool of unmatched agents the previous inequality cannot be strict; otherwise a superior market option $\delta\gamma\overline{v}_{st+1} + \delta(1-\gamma)\overline{x}_{st+1}$ would be attainable at t, contradicting that $\overline{x}_{st}$ is the upper bound of the market options at t. We may reason similarly about the other bounds to conclude that

$$\begin{aligned}
\overline{x}_{st} &= \delta\gamma\overline{v}_{st+1} + \delta(1-\gamma)\overline{x}_{st+1} \\
\underline{x}_{st} &= \delta\gamma\underline{v}_{st+1} + \delta(1-\gamma)\underline{x}_{st+1} \\
\overline{x}_{bt} &= \delta\pi_{t+1}\overline{v}_{bt+1} + \delta(1-\pi_{t+1})\overline{x}_{bt+1} \\
\underline{x}_{bt} &= \delta\pi_{t+1}\underline{v}_{bt+1} + \delta(1-\pi_{t+1})\underline{x}_{bt+1}.
\end{aligned} \tag{5}$$

If the market equilibrium in ultimatum strategies is not the unique market equilibrium then, at a market equilibrium in uniform profiles, if sellers (buyers) obtain their best market option $\overline{x}_{st}$ (respectively $\overline{x}_{bt}$) at all t, buyers (sellers) must attain their worst market option $\underline{x}_{bt}$(respectively $\underline{x}_{st}$) at all t. Consider a market profile where the market options of the sellers are $\overline{x}_{st}$, $\overline{x}_{st} > \underline{x}_{st}$ while the options of the buyers are $\underline{x}_{bt}$, $\underline{x}_{bt} < \overline{x}_{bt}$. At such profile buyers cannot be playing ultimatum strategies: By Proposition 2, market options $\overline{x}_{st}$ and $\underline{x}_{bt}$, different from the market options of the market equilibrium in ultimatum strategies, can be market equilibrium options only if the following are subgame perfect equilibrium strategies for each bargaining pair: as proposer, the seller gives the buyer only $\underline{x}_{bt}$ and threatens (credibly) to take the market option upon rejection; on the other side, buyers that act as first proposers offer sellers their continuation value within the pair (because, given that their market option is just $\underline{x}_{bt}$, they cannot credibly threaten to opt out). That is, when the seller (buyer) is selected as first proposer, agents play a subgame perfect equilibrium with immediate agreement in which she obtains the maximum (minimum) payoff in the interval of payoffs of Proposition 2 i). Symmetric behavior must prevail in a market equilibrium where buyers obtain $\overline{x}_{bt}$ while sellers obtain $\underline{x}_{st}$. Thus the extreme expected values can be written in terms of the extreme options as

$$\begin{array}{r} \overline{v}_{st} = \frac{1+\delta}{2} - \underline{x}_{bt} \\ \underline{v}_{bt} = \frac{1-\delta}{2} + \underline{x}_{bt} \\ \underline{v}_{st} = \frac{1-\delta}{2} + \underline{x}_{st} \\ \overline{v}_{bt} = \frac{1+\delta}{2} - \underline{x}_{st}, \end{array} \tag{6}$$

and substituting (6) in (5) we obtain

$$\begin{array}{r} \overline{x}_{st} = \gamma\delta[\frac{1+\delta}{2} - \underline{x}_{bt+1}] + (1-\gamma)\delta\overline{x}_{st+1} \\ \underline{x}_{st} = (1-\delta)\frac{\delta\gamma}{2} + \delta\underline{x}_{st+1} \\ \overline{x}_{bt} = \pi_{t+1}\delta[\frac{1+\delta}{2} - \underline{x}_{st+1}] + (1-\pi_{t+1})\delta\overline{x}_{bt+1} \\ \underline{x}_{bt} = (1-\delta)\frac{\delta\pi_{t+1}}{2} + \delta\underline{x}_{bt+1}. \end{array} \tag{7}$$

Our next result, Lemma 5 states that this system of difference equations has a unique solution that is admissible. The proof is in the Appendix.

Lemma 5 *The unique solution to (7) such that* $0 \leq \underline{x}_{st} < \overline{x}_{st} \leq 1$ *and* $0 \leq \underline{x}_{bt} < \overline{x}_{bt} \leq 1$ *is*

$$\begin{array}{r} \underline{x}_{st} = \frac{\delta\gamma}{2} \\ \underline{x}_{bt} = \frac{\delta(1-\delta)}{2} B_t \\ \overline{x}_{st} = \gamma\delta S_t \\ \overline{x}_{bt} = \delta\frac{1+\delta(1-\gamma)}{2} C_t \end{array} \tag{8}$$

where

$$\begin{array}{l} B_t \equiv \sum_{i=1}^{\infty} \pi_{t+i}\delta^{i-1} \\ S_t \equiv \sum_{k=0}^{\infty} \delta^k(1-\gamma)^k \frac{1}{2}[1-\delta^2(1-\delta)\sum_{i=1}^{\infty} \pi_{t+i}\delta^{i-1}] \\ C_t \equiv \sum_{k=1}^{\infty} \pi_{t+k}\delta^{k-1} \prod_{j=1}^{k-1}(1-\pi_{t+k})^k. \end{array}$$

The following proposition is now immediate.

Proposition 6 *At any market equilibrium, the market options at each* t *must lie in the intervals* $[\underline{x}_{st}, \overline{x}_{st}]$ *and* $[\underline{x}_{bt}, \overline{x}_{bt}]$*, where* $\{\underline{x}_{st}, \overline{x}_{st}, \underline{x}_{bt}, \overline{x}_{bt}\}$ *is given by (8).*

We say that the *sequence of market options is* δ*-bounded* when $\{x_{st}, x_{bt}\}$ satisfies $x_{it} \leq \delta(\delta - x_{jt+1})$ for $i, j = s, b$ and for all t. Our next result, Proposition 7, establishes necessary and sufficient conditions for the existence of market equilibria sustaining each of the payoffs within the bounds set in Proposition 6. This condition is that the sequences of pairs of market options combining the upper bound of one side of the market with the lower bound of the other side are δ -bounded.

Proposition 7 *If the sequences* $\{\overline{x}_{st}, \underline{x}_{bt}\}$ *and* $\{\overline{x}_{bt}, \underline{x}_{st}\}$ *are* δ*-bounded, then for each sequence* $\{x^{\epsilon}_{st}\}$ *, where* $x^{\epsilon}_{st} = \epsilon \underline{x}_{st} + (1-\epsilon)\overline{x}_{st}$*,* $0 \leq \epsilon \leq 1$*, there is a sequence* $\{x_{bt}\}$ *,* $\underline{x}_{bt} \leq x_{bt} \leq \overline{x}_{bt}$*, such that* $\{x^{\epsilon}_{st}, x_{bt}\}$ *are market equilibrium options. Otherwise the market equilibrium in ultimatum strategies is the unique market equilibrium.*

Proof: Consider bargaining games with outside options $\overline{x}_{st}$ and $\underline{x}_{bt}$. Observe that $\overline{x}_{st} \geq \delta \overline{x}_{st+1}$, $\underline{x}_{bt} \geq \delta\ \underline{x}_{bt+1}$. By Proposition 2, with δ-bounded outside options $\{\overline{x}_{st}, \underline{x}_{bt}\}$, all $v_{st} \in [\frac{1-\delta}{2} + \overline{x}_{st}, \frac{1+\delta}{2} - \underline{x}_{bt}]$ and $v_{bt} \in [\frac{1-\delta}{2} + \underline{x}_{bt}, \frac{1+\delta}{2} - \overline{x}_{st}]$ are expected gains attainable in a subgame perfect equilibrium. Pick a profile where $v_{st} = \frac{1+\delta}{2} - \underline{x}_{bt}$ and $v_{bt} = \frac{1-\delta}{2} + \underline{x}_{bt}$, and observe that the market options that arise under this profile are $\overline{x}_{st}$ and $\underline{x}_{bt}$. Symmetrically we may select market profiles that yield options $\overline{x}_{bt}$ to the buyers and $\underline{x}_{bt}$ to the sellers and that are a subgame perfect equilibrium of games with these outside options if and only if $\{\overline{x}_{bt}, \underline{x}_{st}\}$ is δ-bounded. Hence, if $\{\overline{x}_{st}, \underline{x}_{bt}\}$ and $\{\overline{x}_{bt}, \underline{x}_{st}\}$ are δ-bounded they are market equilibrium sequences of options.

Let us now explore how to sustain market equilibria with interior sequences of outside options $\{x_{st}, x_{bt}\}$, $x_{st} \in [\underline{x}_{st}, \overline{x}_{st}]$ and $x_{bt} \in [\underline{x}_{bt}, \overline{x}_{bt}]$, when $\{\overline{x}_{st}, \underline{x}_{bt}\}$ and $\{\overline{x}_{bt}, \underline{x}_{st}\}$ are δ-bounded. By Proposition 2, in a bargaining game with δ-bounded outside options $\{x_{st}, x_{bt}\}$, all $v_{st} \in [\frac{1-\delta}{2} + x_{st}, \frac{1+\delta}{2} - x_{bt}]$ and $v_{bt} \in [\frac{1-\delta}{2} + x_{bt}, \frac{1+\delta}{2} - x_{st}]$ are expected gains attainable in subgame perfect equilibrium. On the other hand, if $\{x_{st}, x_{bt}\}$ is a market equilibrium sequence of options sustained with a profile that yields expected gains v_{st} to all the sellers and v_{bt} to all the buyers, then the outside options are uniquely given by

$$x_{st} = \delta\gamma v_{st+1} + \delta(1-\gamma)x_{st+1} = \delta\gamma \sum_{i=0}^{\infty} v_{st+i}(1-\gamma)^i \delta^i$$

and

$$x_{bt} = \delta\pi_{t+1} v_{bt+1} + \delta(1-\pi_{t+1})x_{bt+1} = \delta \sum_{i=1}^{\infty} v_{bt+i}\pi_{t+i} \prod_{k=1}^{i} (1-\pi_{t+i-k})\delta^i.$$

Consider a sequence of expected values $v^\epsilon_{st} = \underline{v}_{st} + \epsilon(\overline{v}_{st} - \underline{v}_{st})$ and $v^\epsilon_{bt} = 1 - v^\epsilon_{st}$, and write x^ϵ_{st} and x^ϵ_{bt} to denote the associated market options. Since $x^\epsilon_{st} = \epsilon\underline{x}_{st} + (1-\epsilon)\overline{x}_{st}$ is a linear combination of $\underline{x}_{st}$ and $\overline{x}_{st}$, and since x^ϵ_{bt} and x^ϵ_{st} are bounded above by $\overline{x}_{bt}$ and $\overline{x}_{st}$, the necessary inequalities for δ-boundedness of $\{x^\epsilon_{st}, x^\epsilon_{bt}\}$ hold. By Proposition 2 all values $v_{st} \in [\frac{1-\delta}{2} + x^\epsilon_{st}, \frac{1+\delta}{2} - x^\epsilon_{bt}]$ and $v_{bt} \in [\frac{1-\delta}{2} + x^\epsilon_{bt}, \frac{1+\delta}{2} - x^\epsilon_{st}]$ are attainable as subgame perfect equilibrium payoffs in a bargaining game with outside options $\{x^\epsilon_{st}, x^\epsilon_{bt}\}$. Observe that

$$v^\epsilon_{st} + x^\epsilon_{bt} \leq \overline{v}_{st} + \overline{x}_{bt} \leq \frac{1+\delta}{2} \text{ and } v^\epsilon_{bt} + x^\epsilon_{st} \leq \overline{v}_{bt} + \overline{x}_{st} \leq \frac{1+\delta}{2}.$$

Hence $v^\epsilon_{st} \in [\frac{1-\delta}{2} + x^\epsilon_{st}, \frac{1+\delta}{2} - x^\epsilon_{bt}]$ and $v^\epsilon_{bt} \in [\frac{1-\delta}{2} + x^\epsilon_{bt}, \frac{1+\delta}{2} - x^\epsilon_{st}]$. Therefore $\{x^\epsilon_{st}, x^\epsilon_{bt}\}$ is a market equilibrium sequence of options.

To complete the proof let us check that δ-boundedness of both $\{\overline{x}_{st}, \underline{x}_{bt}\}$ and $\{\overline{x}_{bt}, \underline{x}_{st}\}$ is necessary. First note that each of these two sequences in turn must be a market equilibrium sequence of options. If $\{\overline{x}_{st}, \underline{x}_{bt}\}$ is not δ-bounded there is a date $\tau \geq 0$ such that $\overline{x}_{s\tau} + \delta\underline{x}_{b\tau+1} > \delta^2$, and it is easy to check that the inequality at τ is maintained at posterior dates: $\overline{x}_{st} + \delta\underline{x}_{bt+1} > \delta^2$ and $\underline{x}_{bt} + \delta\overline{x}_{st+1} > \delta^2$ for all $t \geq \tau$. By Proposition 2 ultimatum strategies prevail as the unique subgame perfect equilibrium outcome for all pairs that meet at dates $t \geq \tau$, but a uniform market profile in which all traders use ultimatum strategies from τ on cannot sustain $\{\overline{x}_{st}, \underline{x}_{bt}\}^\infty_{t=\tau}$ as market equilibrium options. The same argument rules out a market equilibrium with options $\{\overline{x}_{bt}, \underline{x}_{st}\}$ when this sequence is not δ-bounded. ■

Remark 4 *Market equilibria that yield intermediate sequences of options* $\{x^\epsilon_{st}\}$ *to the sellers have a direct interpretation: they are the result of a uniform profile of strategies where responders obtain more that their market option. Alternatively, sellers might obtain* $\{x^\epsilon_{st}\}$ *because buyers play a non uniform profile.*

Proposition 7 implies that the market equilibrium in ultimatum strategies prevails uniquely even under configurations for which individual bargaining pairs admit multiple subgame perfect equilibria: at these configurations there are multiple subgame perfect equilibrium payoffs for bargaining pairs, but only the market options of the ultimatum strategies are consistent with market equilibrium. Our next result, Proposition 8, characterizes parameter configurations for which the market equilibrium in ultimatum strategies is the unique market equilibrium.

Proposition 8 *For each initial population and entry flow of traders* (z_0, z) *the following holds: i) Given the matching probability of the sellers* γ *there is a* $\delta^*, 0 < \delta^* < 1$, *such that the market equilibrium in ultimatum strategies is the unique market equilibrium if and only if* $\delta < \delta^*$; *and ii) given the discount rate* δ *there is a* $\gamma^*, \gamma^* < 1$, *such that the market equilibrium in ultimatum strategies is the unique market equilibrium if and only if* $\gamma \geq \gamma^*$.

Proof: i) Observe that $\underline{x}_{st}$, $\overline{x}_{st}$, $\underline{x}_{bt}$ and $\overline{x}_{bt}$ are continuous in all parameters. For each configuration $(z_{0,}z, \gamma)$ consider

$$\begin{aligned}
&\lim_{\delta\to 1} \underline{x}_{st} = \tfrac{\gamma}{2} \\
&\lim_{\delta\to 1} \underline{x}_{bt} = 0 \\
&\lim_{\delta\to 1} \overline{x}_{st} = \gamma \\
&\lim_{\delta\to 1} \overline{x}_{bt} = \tfrac{2-\gamma}{2} \sum_{i=1}^{\infty} \pi_{t+i} \prod_{k=1}^{i} (1 - \pi_{t+i-k})
\end{aligned}$$

and observe that all the inequalities for δ-boundedness hold strictly in the limit. By continuity, for all parameter configurations, there is $\delta < 1$ such that all inequalities are preserved. Hence, for each parameter configuration there is a $\delta^* < 1$ such that $\delta > \delta^*$ implies that $[\underline{x}_{st}, \overline{x}_{st}]$ and $[\underline{x}_{bt}, \overline{x}_{bt}]$ are the intervals of market equilibria.

ii) Simply observe that for each parameter configuration with $\delta < 1$, there is a $\gamma^* < 1$, (possibly $\gamma^* > \delta$!), such that $\{\overline{x}_{st}, \underline{x}_{bt}\}$ is not $\delta-$ bounded. Recall that

$$\overline{x}_{st} = \gamma\delta S_t + (1-\gamma)\delta\overline{x}_{st+1} \text{ and } \underline{x}_{bt} = \frac{\delta(1-\delta)}{2} B_t.$$

Observe that

$$\overline{x}_{st} > \frac{\delta\gamma}{2}\left[1 + \delta\left(1 - B_t\right) + \delta^2 B_t\right]$$

implies that

$$\overline{x}_{st} + \delta\underline{x}_{bt+1} > \frac{\delta\gamma}{2}\left[1 + \delta\left(1 - B_t\right) + \delta^2 B_t\right] + \frac{\delta^2(1-\delta)}{2} B_{t+1}$$

and

$$\frac{\delta\gamma}{2}\left[1 + \delta\left(1 - B_t\right) + \delta^2 B_t\right] + \frac{\delta^2(1-\delta)}{2} B_{t+1} \geq \frac{\delta\gamma(1+\delta)}{2}.$$

For each $\delta < 1$, if $\gamma \geq \frac{2\delta}{1+\delta}$ the previous inequalities imply that for all t, $\overline{x}_{st} + \delta\underline{x}_{bt+1} > \frac{\delta\gamma(1+\delta)}{2} \geq \delta^2$. ■

It is now immediate that when players become arbitrarily patient and search frictions vanish it is not necessarily the case that expected prices in a market equilibrium approach 1, the Walrasian price.

Corollary 9 *There exist sequences* $\{\delta_n, \gamma_n\} \to \{1, 1\}$ *and associated market equilibria with market options* $\{x^n_{st}, x^n_{bt}\}$*, where* $x^n_{st} = \frac{\delta_n \gamma_n}{2}$*. Along such sequence of market equilibria the average price of trades at* t *is* $p^n_t = \frac{1}{2}\frac{\delta_n\gamma_n}{2} + \frac{1}{2}(1 - x^n_{bt})$ *and* $\lim_{(\delta_n,\gamma_n)\to(1,1)} p^n_t = p_t$*, where* $p_t \leq \frac{3}{4} < 1$ *for all* t*.*

5 Steady States

In this section we look at the special case of markets where the measures of buyers and sellers remain at a steady state $(1, s_0)$ thanks to identical flows of entry in both sides of the market $s = b$ that exactly match the measure of realized trades (i.e. as in Rubinstein and Wolinsky [1985]). An interpretation is that traders return to the market after each trade.

In a *stationary market equilibrium* all relevant information about the market at each date t can be summarized by δ, $0 < \delta < 1$, and the constant matching probabilities of buyers and sellers π_b and π_s, $0 < \pi_b \leq \pi_s < 1$.

Since the primitives of the market are time independent, the range of (endogenously given) market options is time independent as well. Consequently, the market options of the sellers and the buyers at each t, x_{st} and x_{bt}, must lie in the time independent intervals $[\underline{x}_s, \overline{x}_s]$ and $[\underline{x}_b, \overline{x}_b]$.

Proposition 10 characterizes market equilibria in the special case of stationary markets. Here the extent of multiplicity is greatly reduced: if frictions are not too great, a unique steady state market equilibrium à la Rubinstein and Wolinsky [1985] prevails. The basic intuition is that the shares that players obtain are determined by the value of the outside opportunities, and in a market where trades and entry keep the measure of buyers and sellers constant these outside opportunities offer positive prospects even to the short side of the market.

Proposition 10 *Assume that matches, trades and entry keep the state of the market constant at* $(s_0, 1)$*. For each* δ *,* π_s *and* π_b *there is* $\pi_\delta < 1$*, such that the following holds.*
i) If $\pi_s \leq \pi_\delta$ *there is a continuum of subgame perfect equilibria for each*

bargaining pair. At each t, the market options of the sellers and the buyers lie in the intervals [$\underline{x}_s$, $\overline{x}_s$] and [$\underline{x}_b$, $\overline{x}_b$], where:

$$\begin{array}{ll} \underline{x}_s = \frac{\delta\pi_s}{2} & \overline{x}_s = \frac{\delta\pi_s}{2} \cdot \frac{1+\delta(1-\pi_b)}{(1-\delta)+\delta\pi_s} \\ \underline{x}_b = \frac{\delta\pi_b}{2} & \overline{x}_b = \frac{\delta\pi_b}{2} \cdot \frac{1+\delta(1-\pi_s)}{(1-\delta)+\delta\pi_b}. \end{array} \tag{9}$$

ii) Otherwise each buyer-seller pair bargaining game has a unique subgame perfect equilibrium in which the responder is offered the value of her market option and she accepts. These options are:

$$x_s = \frac{\frac{\delta}{2}\pi_s}{(1-\delta)+\frac{\delta}{2}(\pi_s+\pi_b)} \text{ and } x_b = \frac{\frac{\delta}{2}\pi_b}{(1-\delta)+\frac{\delta}{2}(\pi_s+\pi_b)}. \tag{10}$$

Proof: See Appendix. ■

The present characterization does not rely on a specific functional form for the matching technology, and consequently a variety of asymptotic outcomes can be explored.

Observe that

$$\lim_{\delta\to 1} x_s = \frac{\pi_s}{\pi_b+\pi_s}.$$

If we let $\pi_b = \pi_s s_0$, and assume that for each state of the market the probability that sellers find a match is unrelated to the cost of bargaining δ, then we obtain that $\lim_{\delta\to 1} x_s = \frac{1}{s_0+1}$ and $\lim_{\delta\to 1} x_b = \frac{s_0}{s_0+1}$. That is the outcome highlighted by Rubinstein and Wolinsky [1985]: each side of the market appropriates a surplus share equal to the relative measure of the other side.

Other asymptotic outcomes are possible, though, if the matching probabilities are related to the delay caused by searching for a new partner in the market. For instance, agents might be constrained to contacting at most one potential partner per period, and only a portion of potential partners are available. In this case, δ and π_s are no longer independent. As an example, let $\pi_b = \delta^{\frac{k}{s_0}}$ and $\pi_s = \delta^k$. Now

$$x_s = \frac{\frac{\delta^{k+1}}{2}}{(1-\delta)+\frac{\delta^{k+1}}{2}\left(\delta^{\frac{k+1}{1+s_0}}\right)} \text{ and } x_b = x_s\delta^{\frac{k}{1+s_0}}.$$

When the market approaches a frictionless market (in the limit as $\delta \to 1$) the market options approach $\lim_{\delta\to 1} x_s = \lim_{\delta\to 1} x_b = \frac{1}{2}$. In this example, the relative scarcity of sellers in the market is offset by the fact that search costs are always higher than the cost of delay inside a bargaining pair. Therefore, the relative bargaining power of both buyers and sellers is symmetric and the 50-50 split prevails (uniquely) regardless of the relative sizes of the two populations.

6 Conclusions

Two-sided and time-varying outside options have substantive consequences in bargaining games: Our predictions about isolated bargaining pairs are substantially less sharp than what is usual since the uniqueness of subgame perfect equilibrium - that prevails in bargaining without options - no longer holds. Moreover, in the context of a market, the outside options are endogenous. The greater complexity of the bargaining game and the endogenous nature of the outside options are not an obstacle for a complete and explicit characterization of market equilibria, even in non-stationary environments. Here, the outside option principle is no longer at work, all outside options are relevant and the range of prices depends fundamentally on the state of the market. When a unique non-stationary market equilibrium prevails, it is sustained by profiles of ultimatum strategies and its limit outcome as frictions vanish is Walrasian. However, since the bargaining game admits a continuum of subgame perfect equilibria, a continuum of market equilibria may prevail as well; and convergence to Walrasian prices is not assured. Therefore, in markets where buyer-seller pairs play alternating offer games with two-sided freedom to search, limit Walrasian outcomes when frictions vanish cannot be assured unless the restriction to ultimatum strategies is exogenously imposed.

Further research is needed to extend the present work to markets with heterogeneous buyers and sellers. In the present environment with homogeneous traders, even at non-Walrasian prices, efficiency (constrained to the feasibility imposed by the matching process) is ensured as long as all realized matches reach immediate agreement. As this no longer holds in markets with heterogeneous buyers and sellers, it is an open question whether inefficient market equilibria can be supported.

Exploration of markets with heterogeneous agents should also illuminate our intuition about stationary markets. As long as entry is exogenous, markets with homogeneous traders approach a stationary state only if equal numbers of buyers and sellers enter. A meaningful model of steady states in markets will have to address endogenous entry as well as endogenous exit, demanding that heterogeneity plays an explicit role.

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A Appendix

Proof of Lemma 3:

Let $a_t = x_{bt} + x_{st}$. It is immediate that $\{a_t\}$ must solve 3.

Since all matches reach trade the measures of active traders at each date and the probabilities that searchers find a partner are uniquely determined from the exogenous parameters of the market (z_0, z, γ), and so is the sequence $\{\Delta_{t+1}\}$. By forward substitution of the sequence $\{\Delta_{t+1}\}$ we obtain that a_t must satisfy

$$a_t = \sum_{\tau=0}^{\infty} \delta^{\tau+1}(1 - \Delta_{t+1+\tau})\Psi_{t+\tau} \tag{11}$$

where

$$\Psi_t = 1, \quad \text{and for } k \geq 1, \quad \Psi_{t+k} = \Psi_{t+k-1}\Delta_{t+k}.$$

Since $\Delta_t < 1 - \frac{\gamma}{2}$, and $\Psi_{t+k} < \gamma^{k-1}$, the right hand side of (11) converges and

$$a_t < \sum_{\tau=0}^{\infty} \delta^{\tau+1}(1 - \frac{\gamma}{2})\gamma^{\tau-1} \equiv A_0$$

for all t. Moreover,

$$a_t < \delta\left((1-\Delta_{t+1}) + \Delta_{t+1}A_0\right)$$

and hence

$$a_t < \frac{\delta}{2}\left(\gamma + (2-\gamma)\,A_0\right) \equiv A_1.$$

Observe that the sequence

$$A_k = \frac{\delta}{2}\left(\gamma + (2-\gamma)\,A_{k-1}\right)$$

is such that for all t and k, $a_t < A_{k+1}$, and

$$a_t \leq \lim_{k\to\infty} A_k = \frac{\delta\gamma}{2\,(1-\delta)+\delta\gamma} < \delta.$$

Therefore (11) yields a unique solution to (3) and $a_t \in (0,\delta)$ for all t. It is now straightforward that the unique solution to (1) must satisfy (2). ■

Proof of Lemma 5:

(In what follows recall that $\gamma, \delta \in (0,1)$, $\pi_{t+1} \in (0,\gamma)$, $\pi_{t+1} < \pi_t$, and $\pi_t \to 0$.) First note that $\underline{x}_{st}$ and $\underline{x}_{bt}$ must solve independent linear difference equations. Any solution $\underline{x}_{st}$ and $\underline{x}_{bt}$ is of the form

$$\underline{x}_{st} = K_s(\tfrac{1}{\delta})^t + x_{st} \quad \underline{x}_{bt} = K_b(\tfrac{1}{\delta})^t + x_{bt}$$

where x_{st} and x_{bt} are particular solutions; and no solution with $K_s, K_b \neq 0$ is bounded. It is easy to verify that the values given in (8) are a particular solution and satisfy $\underline{x}_{st}, \underline{x}_{bt} \in (0, \frac{1}{2})$ for all t. Hence they are the unique admissible values for $\underline{x}_{st}$ and $\underline{x}_{bt}$.

Substituting these particular solutions into (7) yields

$$\begin{aligned} \overline{x}_{st} &= \gamma\delta[\tfrac{1+\delta}{2} - \tfrac{\delta(1-\delta)}{2}B_{t+1}] + (1-\gamma)\delta\overline{x}_{st+1} \\ \overline{x}_{bt} &= \pi_{t+1}\delta[\tfrac{1+\delta(1-\gamma)}{2}] + (1-\pi_{t+1})\delta\overline{x}_{bt+1} \end{aligned} \tag{12}$$

It is now an exercise to verify that (8) gives particular solutions that lie in $(0,1)$. The uniqueness of a bounded solution $\overline{x}_{st}$ follows since any solution must be of the form

$$\overline{x}_{st} = K(\frac{1}{\delta(1-\gamma)})^t + x_t.$$

To check that no other solution $\overline{x}_{bt}$ is bounded consider, for each t, the sequence $\left\{\underline{b}_t^k, \overline{b}_t^k\right\}_{k=0}^{\infty}$ where $\underline{b}_t^0 = 0$, $\overline{b}_t^0 = 1$,

$$\underline{b}_t^k = \pi_{t+1}\delta[\frac{1+\delta(1-\gamma)}{2}] + (1-\pi_{t+1})\delta\underline{b}_t^{k-1}$$

$$\overline{b}_t^k = \pi_{t+1}\delta[\frac{1+\delta(1-\gamma)}{2}] + (1-\pi_{t+1})\delta\overline{b}_t^{k-1}$$

and observe that $\underline{b}_t^k \leq \overline{x}_{bt} \leq \overline{b}_t^k$, $\underline{b}_t^k > \underline{b}_t^{k-1}$, $\overline{b}_t^k < \overline{b}_t^{k-1}$, and $\lim_{k\to\infty} \underline{b}_t^k = \lim_{k\to\infty} \overline{b}_t^k = \overline{x}_{bt}$.

Finally we check that $\underline{x}_{st} < \overline{x}_{st}$ and $\underline{x}_{bt} < \overline{x}_{bt}$. The first inequality is immediate. To check the second it suffices to note that

$$\underline{x}_{bt} = \frac{\delta(1-\delta)}{2}\sum_{i=1}^{\infty}\pi_{t+i}\delta^{i-1} < \frac{\delta(1-\delta)}{2}\pi_{t+1}\sum_{i=1}^{\infty}\delta^{i-1} = \frac{\pi_{t+1}\delta}{2}$$

and $\overline{x}_{bt} = \pi_{t+1}\delta[\frac{1+\delta(1-\gamma)}{2}] + (1-\pi_{t+1})\delta\overline{x}_{bt+1} > \frac{\pi_{t+1}\delta}{2}$. ■

Proof of Proposition 10:

For each buyer-seller pair the bargaining game starting in period t yields subgame perfect equilibrium values v_{st} and v_{bt}, respectively in $[\underline{v}_{st}, \overline{v}_{st}]$ and $[\underline{v}_{bt}, \overline{v}_{bt}]$. Observe that, since the matching probabilities are constant for buyers and sellers, this intervals do not depend on t. The market options must lie in $[\underline{x}_{st}, \overline{x}_{st}]$ and $[\underline{x}_{bt}, \overline{x}_{bt}]$ that must also be independent of t. Therefore

$$\begin{array}{ll}\underline{x}_s = \delta\pi_s\underline{v}_s + \delta(1-\pi_s)\underline{x}_s & \overline{x}_s = \delta\pi_s\overline{v}_s + \delta(1-\pi_s)\overline{x}_s \\ \underline{x}_b = \delta\pi_b\underline{v}_b + \delta(1-\pi_b)\underline{x}_b & \overline{x}_b = \delta\pi_b\overline{v}_b + \delta(1-\pi_b)\overline{x}_b.\end{array}$$

In order to characterize the equilibria that yield the supremum (infimum) value to sellers (buyers) there is no loss of generality if we restrict attention to uniform profiles in which all pairs trade. For this class of strategy profiles, all buyers and sellers expect the same value if they return to the market; that is, they have the same market option. By Lemma 1 there is always an equilibrium in which the proposer gives the responder just the outside option. Since both players act as first proposer (responder) with probability $\frac{1}{2}$, we have that:

$$\begin{array}{ll}\underline{v}_s = \frac{1}{2}[(1-\delta) + 2\underline{x}_s] & \overline{v}_s = \frac{1}{2}[(1+\delta) - 2\underline{x}_b] \\ \underline{v}_b = \frac{1}{2}[(1-\delta) + 2\underline{x}_b] & \overline{v}_b = \frac{1}{2}[(1+\delta) - 2\underline{x}_s].\end{array}$$

Substitution yields (9).

The bounds set in (9) characterize the set of subgame perfect equilibria provided that: $\overline{x}_s + \delta \underline{x}_b \leq \delta^2$, $\overline{x}_b + \delta\ \underline{x}_s \leq \delta^2$, $\underline{x}_s + \delta \overline{x}_b \leq \delta^2$ and $\underline{x}_b + \delta \overline{x}_s \leq \delta^2$. It is easy to check that $\overline{x}_s + \delta \underline{x}_b \leq \delta^2$ is the most stringent of these inequalities. Thus, as soon as $\overline{x}_s + \delta \underline{x}_b > \delta^2$ we have a unique subgame perfect equilibrium in which proposers offer the responder just the (unique!) outside option. Observe that $\overline{x}_s + \delta \underline{x}_b > \delta^2$ if and only if

$$\frac{\delta \pi_s}{(1-\delta) + \delta \pi_s} \geq \frac{\delta (2 - \pi_b)}{1 + \delta (1 - \pi_b)}$$

and note that $\frac{\delta \pi_\delta}{(1-\delta)+\delta\pi_\delta} = \frac{\delta(2-\pi_b)}{1+\delta(1-\pi_b)}$ yields $\pi_\delta < 1$ for all π_b and all $\delta \leq 1$. Hence $\overline{x}_s + \delta\ \underline{x}_b > \delta^2$ if and only if $\pi_s > \pi_\delta$. This proves i).

If each buyer-seller pair plays a bargaining game that yields unique subgame perfect equilibrium values v_s and v_b, then x_s and x_b must be given by $x_s = \delta \pi_s v_s \sum_{t=0}^{\infty} (1-\pi_s)^t \delta^t = \frac{\delta \pi_s v_s}{(1-\delta)+\delta\pi_s}$ and $x_b = \delta \pi_b v_b \sum_{t=0}^{\infty} (1-\pi_b)^t \delta^t = \frac{\delta \pi_b v_b}{(1-\delta)+\delta\pi_b}$.

Now if the endogenously given x_s and x_b are such that $x_s + \delta x_b > \delta^2$ and $x_b + \delta x_s > \delta^2$, then by Proposition 2 ii) we must have $v_s = \frac{1}{2}(1-x_b) + \frac{1}{2}x_s$ and $v_b = \frac{1}{2}(1-x_s) + \frac{1}{2}x_b$.

Thus, x_s and x_b must solve $x_s = \alpha_s(1-x_b)$, $x_b = \alpha_b(1-x_s)$, with

$$\alpha_s = \frac{\frac{\delta}{2}\pi_s}{(1-\delta)+\delta\pi_s} \text{ and } \alpha_b = \frac{\frac{\delta}{2}\pi_b}{(1-\delta)+\delta\pi_b}.$$

That yields (10) and completes the proof of ii). ■