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The war of attrition with incomplete information

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Abstract

We present a continuous-time model of the war of attrition with exponential discounting and with two-sided incomplete information. We provide a full characterization of the Bayesian Equilibria of this game, without restricting strategies to be differentiable.

Keywords: War of attrition; Concession games

1. Introduction

In this paper we present an analysis of the war of attrition in continuous time with two-sided incomplete information about reservation values. In this game two players, each favoring (a different) one of two alternatives, must choose a time at which to concede in case the other has not done so yet. The payoff from any individually rational outcome decreases over time, but at any time a player prefers that her opponent concede rather than conceding herself. This game, originally proposed by Maynard Smith (1974) to study patterns of animal behavior, is useful in the study of a wide variety of conflict situations. Examples in the literature range from price wars and exit in oligopolistic markets (Fudenberg and Tirole, 1986; Ghemawat and Nalebuff, 1985; Kreps and Wilson, 1982) to patent races (Fudenberg et al., 1983), public good provision (Bliss and Nalebuff, 1984), and bargaining (Ordover and Rubinstein, 1982; Osborne, 1985; Chatterjee and Samuelson, 1987).

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Hendricks et al. (1988) present a very general analysis of the war of attrition with *complete* information. The set of Nash Equilibria in such games is large: apart from the asymmetric outcomes in which one party concedes immediately in equilibrium, a continuum of equilibria also arise in which players randomize in choosing the date of concession. The first analysis of the game with incomplete information is due to Bishop et al. (1978). It was taken up later by Riley (1980), Milgrom and Weber (1985) and Nalebuff and Riley (1985), in addition to the papers previously mentioned. The general conclusion of the literature can be roughly paraphrased as: "There are a continuum of equilibria characterized by a system of ordinary differential equations. Uniqueness may be achieved by perturbing the game, imposing that for a positive measure of types it is a dominant strategy not to concede." This perturbation may take the form of assuming that some players are irrational, as in Nalebuff and Riley (1985), or by changing the payoffs to disagreement with a small probability, as in Fudenberg and Tirole (1986).

Extending these results, we set up a model which, unlike the standard version of the war of attrition, that models impatience by an additive linear cost of delay, uses exponential discounting. For this model we provide a self-contained full characterization of the set of equilibria. This approach makes it possible to incorporate naturally types that prefer disagreement to concession, since the disagreement payoff is zero (as opposed to minus infinity). This way situations that have a unique equilibrium arise as an integral part of the model. Perhaps even more importantly, this formulation is much more appropriate to use in generalizations of the war of attrition: to more than two players (Ponsati and Sákovics, 1995), to bargaining over many issues (Ponsati, 1992) or to bargaining over a finite number of alternatives (Ponsati and Sákovics, 1992). For uniqueness of the equilibria in 'subgames' of these generalizations allows inductive characterizations of the equilibria.

There are two main classes of types in our model: weak types, i.e. players that prefer concession to disagreement, and tough types, i.e. players that would rather never agree than concede. In the unique equilibrium tough types never concede while, provided that they are weak, both players distribute their concessions across time. These latter strategies are characterized by a pair of differential equations. When the prior distribution of types is uniform over some finite interval, the equilibrium concession rates are such that at all times the conditional probability that Player i is tough is the same as the probability that Player j is tough. In general, a player more likely to be tough receives a higher expected payoff. When it is not known that exactly one of the players is weak with probability one, reaching agreement takes (possibly infinite) time because a strategy in which all weak types of a given player concede by some time cannot be supported as an equilibrium – waiting an instant to convince the opponent of their toughness always increases the expected payoff of weak players.

2. The model

Consider two agents disputing over two alternatives A and B. Player a favours A, Player b favours B. Let us denote a generic alternative by X (similarly, denote a generic player by x). Then Player i's ($i = \mathrm{a}, \mathrm{b}$) static preferences can be described as follows:[1]

$$u_i(X, s) = \begin{cases} 1 - s\,, & \text{if } i = x\,,\\ -s\,, & \text{if } i \neq x\,,\end{cases}$$

where s, his type, is a privately known parameter (reservation value).

The game is played in continuous time, starting at $t = 0$. The players have a unique action available to them, though they can freely choose its timing, they can *yield*. Player a (b) starts out proposing alternative A (B) and this situation persists until she yields, at which time B (A) is implemented. (In the case that both players concede at the same time, they use a lottery to decide the outcome. The only requirement we impose is that the lottery cannot assign probability one to either alternative.) Thus, a strategy for player i, σ_i, is a (measurable) function from his type to the (possibly infinite) time of his concession.

Players are assumed to be impatient. Their impatience is modeled by a common discount factor, normalized to be e^{-1} per unit of time. Thus, Player i receives (intertemporal) utility $U_i(X, s, t) = u_i(X, s)\, e^{-t}$ if alternative X is reached at time t.[2] Perpetual conflict gives 0 to both players.

The players entertain beliefs about each other's type. We assume that these a priori beliefs are common knowledge and they can be represented by a probability distribution F_i, with positive density f_i, over the interval $[i_L, i_H]$, $f_i \in C^\infty$. We impose two assumptions about these intervals: (i) $i_L < 0$; (ii) $i_H \leq 1$. That is, with positive probability both players are of a type who derives a positive utility even if the opponent's alternative prevails (weak type); and there are no types of either player who derive negative utility even if their preferred alternative is agreed upon. Finally, we assume that the type of player a is independent from player b's type.

Given a strategy profile σ let $(X(\sigma(s, z)), t(\sigma(s, z)))$ be the outcome generated by σ if types are s and z, and let $V_i^s(\sigma)$ denote the expected payoff to Player i of type s given σ:

$$V_i^s(\sigma) = \int_{[j_L, j_H]} U_i(X(\sigma(s, z)), s, t(\sigma(s, z)))\, \mathrm{d}F_j(z)\,.$$

[1] We use the above preferences for simplicity. We actually only need the following restrictions on preferences (assume we deal with Player a): $u(B, s) \geq u(\text{Disagreement})$ iff s is non-positive; $u(A, s) > u(B, s)$ for all s; $\partial u(X, s)/\partial s < 0$; and $u(B, s') - u(B, s) \geq u(A, s') - u(A, s)$ for $s' \geq s$.

[2] Again, the assumption on time preferences is for simplicity, it would suffice to have $U_i(X, s, t) = u_i(X, s)\, \phi(t)$ with ϕ continuously differentiable, strictly decreasing and $\lim_{t\to\infty} \phi(t) = 0$.

Then σ is a *Bayesian Equilibrium* (BE) if and only if for all (s, z) and $i = a, b$: $V_i^s(\sigma) \geq V_i^s(\sigma_i', \sigma_{-i})$ for all σ_i'.

3. The solution

In the following proposition we characterize the entire set of Bayesian equilibrium outcomes of our game.

Proposition 1. *The set of BE outcomes is characterized by the following*:

(i) *If* $i_H \leq 0$ *and* $0 < j_H$, *then, almost surely, i (and only i) concedes at* 0 (and vice versa).

(ii) *If* $0 < i_H$ *and* $0 < j_H$, *then strategies are such that* $\sigma_i(s) = t$ *if and only if* $\varepsilon(t) = s$, *and* $\sigma_j(z) = t$ *if and only if* $\zeta(t) = z$, *where* $\varepsilon : (0, \infty) \to (i_L, 0]$ *and* $\zeta : (0, \infty) \to (j_L, 0]$ *are increasing, differentiable functions that uniquely solve the following system of differential equations*:

$$
\begin{aligned}
-\zeta'(t) f_j(\zeta(t)) &= [1 - F_j(\zeta(t))]\varepsilon(t) , \\
-\varepsilon'(t) f_i(\varepsilon(t)) &= [1 - F_i(\varepsilon(t))]\zeta(t) ,
\end{aligned}
\tag{1}
$$

with $\lim_{t\to\infty}(\varepsilon(t), \zeta(t)) = (0, 0)$ *and* $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$.

(iii) *If* $i_H \leq 0$ *and* $j_H \leq 0$, *then there is a continuum of BE characterized by the solutions to* (1), *such that* $(\lim_{t\to 0} \varepsilon(t) - i_L)(\lim_{t\to 0} \zeta(t) - j_L) = 0$. *These include two equilibria where, almost surely, exactly one player concedes at zero. In the other equilibria, if* $i_H < 0$ *and* $j_H < 0$, $\lim_{t\to T}(\varepsilon(t), \zeta(t)) = (i_H, j_H)$ *for some* $T \leq \infty$; *and if* $i_H = 0$, $\lim_{t\to\infty}(\varepsilon(t), \zeta(t)) = (\eta, \varphi)$, *for* $i_L \leq \eta \leq i_H$, $j_L \leq \varphi \leq j_H$, $(\eta - i_H)(\varphi - j_H) = 0$.

Corollary. *Imposing sequential rationality does not refine the set of equilibria. That is, the set of Perfect Bayesian Equilibrium outcomes is the same as the one characterized in Proposition* 1.

Proof (of the Corollary). Note that in the equilibria with concessions distributed over the real line, there are no off-the-equilibrium-path events since any move terminates the game and every move (made at a finite time) has positive probability. The other equilibria are also perfect, for every type s of the player who is not supposed to concede there is a $\rho(s) > 0$, such that if the game has not ended by t and he believes that his opponent will concede almost surely before $t + \rho(s)$ he will never concede. □

Note that positive (that is, tough) types prefer perpetual conflict to any outcome that gives them their less preferred alternative and therefore they will never yield in equilibrium. Thus, if one of the players is believed to be tough with positive probability, it is credible for him to claim that he will never yield. Therefore, as in the battle of the sexes, if his opponent is known to be willing to yield [case (i)], he can take advantage of her. If both can be unwilling to yield, we get the classical war-of-attrition behavior, the players try to screen each other's type by prolonging the game and thus imposing a delay cost on their opponent (as well as on themselves). If both players are known to prefer yielding to perpetual conflict, we obtain a continuum of the wearing-down strategies mentioned above, and two limit cases in which one of the players yields immediately. The latter are possible because now there is nothing that can convince a player that the other will never concede. Therefore if either of them believes after any time without concession that the other will soon concede, it is optimal for her to hold on, but then her opponent is best off conceding as soon as he can (at zero). Case (iii) is the equivalent of the linear cost case with a bounded support of types, if $i_H < 0$ and $j_H < 0$, or the limiting case as the support of types becomes unbounded, if $i_H = 0$ as in Nalebuff and Riley (1985).

Before getting to the proof of Proposition 1, we introduce some notation and intermediate results.[3] Let $H_i^\sigma(t)$ denote the probability that i gives in no later than t according to σ. We will write $H_i(t)$ if no confusion arises. Correspondingly, $H_i(\infty)$ will denote the probability that i will ever concede.

Lemma 0 (Positive types out). *It is without loss of generality to assume that types $s > 0$ never concede in equilibrium.*

Proof. Since a player with positive type obtains negative utility upon concession, while the disagreement outcome gives him zero, he would only concede at times by which the other player would have already conceded with probability one. That is, off the equilibrium path. □

Based on Lemma 0, from now on we restrict our attention to the strategies of non-positive types.

Lemma 1 (Simultaneous gaps). *In any BE, along the equilibrium path, strategies are such that if the probability that i concedes in the interval $(t, t+\delta]$ is nil, then, almost surely, j does not concede in that interval either, that is, $H_j(t) = H_j(t+\delta)$.*

Proof. If a player knows that her opponent does not yield in the interval $(t, t+\delta]$,

[3] Some very similar results to many of these lemmas have already been proven in the literature (cf. Nalebuff and Riley, 1985; Fudenberg and Tirole, 1986), however, for the sake of completeness we present a complete treatment here.

then making a concession at any t' in $(t, t+\delta]$ cannot maximize her payoff: since payoffs are discounted, any player who obtains a positive payoff from concession will benefit if she concedes a bit earlier. □

Lemma 1 is instrumental in later proofs. It is also very intuitive: the only reason for a player to hold out is because she expects that the opponent will give in 'soon'. If it is known that he will not, it is better to concede as early as possible. Lemma 2, implied by Lemma 1, states that at no time other than 0 is there a mass point of concessions by either player.

Lemma 2 (Continuity). *For all* $0<t<T$, $H_i(t)$ *satisfies a Lipschitz condition with* $K=-j_L\,e^{T+1}$. *(And therefore it is also continuous.)*

Proof. Assume that $H_i(t)$ does not satisfy a Lipschitz condition with $K=-j_L\,e^{T+1}$. That is, there is a $t>0$, such that there exists δ in $(0,1)$ such that for all t' in $(t-\delta, t+\delta)$, $H_i(t+\delta)-H_i(t')>-(t+\delta-t')\,j_L\,e^{T+1}$. Assume moreover, that j intends to concede at t'. The gain in undiscounted expected payoff he obtains if he waits and concedes at $t+\delta$ is not less than

$$\frac{H_i(t+\delta)-H_i(t')}{1-H_i(t')}\,e^{-t-\delta}$$

which by assumption is strictly greater than $-(t+\delta-t')j_L$. The loss that he incurs because his payoff is discounted is $-s(e^{-t+\delta-e^{-t'}})$ which is strictly less than $-(t+\delta-t')j_L$. Hence concession at $t+\delta$ yields a higher payoff than a concession at t' to all types, and thus, with probability 1, j does not concede in $(t', t+\delta)$. But then Lemma 1 implies that in $(t', t+\delta)$ i does not concede either, contradicting that $H_i(\cdot)$ is increasing at t. □

Remark. Note that this argument does not hold for $t=0$.

Let $T_i=\min\{t$, such that $H_i(t)=H_i(\infty)\}$. That is, T_i is the earliest time by which i gives in with probability one, conditional on ever giving in.

We next show, in Lemma 3, that T_i is the same for both players and, in Lemma 4, that $H_i(\cdot)$ must be strictly increasing in $(0, T_i)$.

Lemma 3 (Symmetric spread). *In every BE, if both* T_a *and* T_b *are positive, then they are equal.*

Proof. Follows from Lemma 1. □

Lemma 4. (Strict monotonicity). *In every BE, unless the game ends at zero, the* *probability that either player concedes is positive in any time interval up to T_i. That is, $H_i(t-\delta) < H_i(t)$ for $0 < \delta < t \leq T_i$.*

Proof. Assume to the contrary that with probability one i will not give in in the interval $(t, t+\delta]$. Consequently, by Lemma 1 j will not give in either. Let δ^* be the supremum of such δ. For any $t < T_i$, $\delta^* < \infty$, so for any $\tau > t + \delta^*$ the probability of both players conceding in $[t+\delta^*, \tau)$ is positive. For τ sufficiently close to $t + \delta^*$ both players prefer to concede at any t' in $(t, t+\delta^*/2)$ to conceding at τ, since by Lemma 2 the probability that the opponent concedes in $[t+\delta^*, \tau)$ is not bounded away from zero. A contradiction. □

Lemma 5 rules out simultaneous concessions at 0.

Lemma 5 (Immediate concessions). *In no BE do both players concede at 0 with positive probability. That is, $H_a(0) \cdot H_b(0) = 0$.*

Proof. Let a concede at 0 with probability $P > 0$. If b concedes at 0 also, he gets a payoff of $(\alpha - s)P - s(1-P)$, where $0 < \alpha < 1$ is the probability of his alternative prevailing in the case of a lottery, while if he concedes at $\delta > 0$, he gets at least $(1-s)P - s(1-P)\,e^{-\delta}$. Since for any s there is some $\delta(s) > 0$ such that the second payoff dominates the first one, he is better off by holding out for a while. □

The following lemma sets up the appropriate objective function that players maximize:

Lemma 6 (Best response mapping). *In a BE strategy profile σ,*

$$\sigma_i(s) \in \operatorname*{argmax}_{t \geq 0} \int_{[0,t]} (1-s)\,e^{-\tau}\,dH_j(\tau) - (1 - H_j(t))s\,e^{-t}\,.$$

Proof. If i concedes at $t > 0$, then his expected utility is given by

$$W_i(s,t) = \int_{[0,t]} (1-s)\,e^{-\tau}\,dH_j(\tau) - (1 - H_j(t))s\,e^{-t}\,.$$

We thus only need to check what happens at zero. The expected payoff of Player i if he gives in at zero is $(\alpha - s)\;H_j(0) - s(1 - H_j(0))$, where $0 < \alpha < 1$ is the probability of his alternative prevailing in the case of a lottery. Note that this only differs from the above payoff in that α substitutes 1 in the first term. By Lemma 5, if $H_j(0) > 0$, i will always prefer not to yield at zero, while if $H_j(0) = 0$, then this difference does not matter. □

Next, Lemma 7 shows that agreeing with intuition, a player for whom the relative difference between the alternatives is smaller is willing to incur less delay cost in order to force a favorable outcome.

Lemma 7 (Strategies monotone in type). *Every BE is such that for all i, $s<s'$ implies that $\sigma_i(s)\le\sigma_i(s')$.*

Proof. Recall that $W_i(s,t)$ denotes the expected payoff of Player i of type s conceding at time t in a BE. Let $s<s'$ and $\sigma_i(s)=t$ and $\sigma_i(s')=t'$. Then it must be that $W_i(s,t')\le W_i(s,t)$ and $W_i(s',t)\le W_i(s',t')$ and, consequently, $W_i(s,t')-W_i(s,t)\le W_i(s',t')-W_i(s',t)$. Let us write out this inequality:

$$\begin{aligned}(1-s)\int_{[t,t')} e^{-v}\,dH_j(v) - s[(1-H_j(t'))(e^{-t'}-e^{-t}) + (H_j(t)\\ - H_j(t'))\,e^{-t}]\\ \le (1-s')\int_{[t,t')} e^{-v}\,dH_j(v) - s'[(1-H_j(t'))(e^{-t'}-e^{-t})\\ + (H_j(t)-H_j(t'))\,e^{-t}]\,,\end{aligned}$$

or

$$(1-H_j(t'))(e^{-t'}-e^{-t}) + (H_j(t)-H_j(t'))\,e^{-t} \le \int_{[t',t)} e^{-v}\,dH_j(v)\,.$$

Assume $t>t'$. Then, by Lemma 4,

$$\begin{aligned}\int_{[t',t)} e^{-v}\,dH_j(v) &< (H_j(t)-H_j(t'))\,e^{-t'}\\ &= (H_j(t)-H_j(t'))(e^{-t'}-e^{-t}) + (H_j(t)-H_j(t'))\,e^{-t}\,.\end{aligned}$$

Since $(e^{-t'}-e^{-t})>0$, the fact that $H_j(t)\le 1$ implies a contradiction. Therefore $t\le t'$. □

Next we show that the optimal dates of concession for each type yield differentiable concession probabilities (Lemma 8), and thus for dates in $(0,T_i)$ a first-order condition must be satisfied (Lemma 9).

Lemma 8 (Differentiability). *Every BE yields differentiable concession probabilities. That is, there is a continuous function $h_i(\cdot)$, such that for all $t\in(0,T_i)$,*

$$H_i(t) = H_i(0) + \int_{(0,t)} h_i(\tau)\,d\tau\,.$$

Proof. Observe, that $H_j(t)$ is differentiable at if and only if $W_i(s,t)$ is. Pick t^* in $(0,T_j)$ and assume that $H_j(t)$ is not differentiable at t^*. Since by Lemmas 4 and 7 some type concedes at every point in time in equilibrium, there is an s such that $W_i(s, t)$ has a maximum at t^*.

Claim. $W_i(s, t)$ is single peaked in t.

We will prove that the set interior local minima of $W_i(s, t)$ is empty. Suppose $W_i(s, t)$ has some interior local minimum, say at t'. We are going to show that t' cannot be an argmax of $W_i(\cdot, t)$ for any s', contradicting that at every point in time some type concedes. Notice that $W_i(s, t)$ can be written as $a(t) - sb(t)$. Since t' is assumed to be a local argmin we have that for all t'' in a neighborhood of t'

$$a(t'') - sb(t'') \geq a(t') - sb(t') .$$

If t' were the argmax from some s' then we would have that for all the t'' for which the previous inequality holds,

$$a(t'') - s'b(t'') \leq a(t') - s'b(t') .$$

Summing the two inequalities and simplifying, we obtain that

$$(s' - s)b(t'') \geq (s' - s)b(t') ,$$

which implies that $b(t)$ has to have a local extremum at t'. Assume it is a maximum. Then by the first inequality (recall that $s < 0$): $a(t'') \geq a(t')$. Since t'' can take values both smaller and greater than t' and since $a(t)$ is strictly monotone this cannot be. Finally assume that the local extremum of $b(t)$ at t' is a minimum. Then by the second inequality $a(t') \geq a(t'')$ and we again reach a contradiction. Consequently, the set of interior local minima of $W_i(s, t)$ is indeed empty. Therefore, since $W_i(s, t)$ is continuous in t, it is single peaked in t.

Since by assumption, W_i is non-differentiable in t at t^* and it has a (global) maximum at t^*, there must exist positive sequences δ_n and μ_n converging to zero such that:

$$\lim_{n\to\infty} \frac{a(t^*) - sb(t^*) - a(t^* - \delta_n) + sb(t^* - \delta_n)}{\delta_n} \geq 0 ,$$

and

$$\lim_{n\to\infty} \frac{a(t^*) - sb(t^*) - a(t^* + \mu_n) + sb(t^* + \mu_n)}{\mu_n} \geq 0 ,$$

with at least one strict inequality. W.l.o.g. let

$$\lim_{n\to\infty} \frac{a(t^*) - sb(t^*) - a(t^* - \delta_n) + sb(t^* - \delta n)}{\delta_n} > 0 .$$

Note that the sequence

$$\frac{b(t^*) - b(t^* - \delta_n)}{\delta_n}$$

is bounded by Lemma 2 and the fact that e^{-t} is bounded and differentiable. Therefore there exists $\nu > 0$ such that for all s' in $(s - \nu, s + \nu)$, the inequality remains. Moreover, the Lipschitz condition implies that there is a positive sequence converging to 0, γ_n, such that

$$c = \lim_{n \to \infty} \frac{a(t^*) - s'b(t^*) - a(t^* + \gamma_n) + s'b(t^* + \gamma_n)}{\gamma_n}$$

exists. By continuity and single-peakedness, $c > 0$ for s' in either $(s - \nu, s)$ or $(s, s + \nu)$. Hence t^* is a maximum for $W_i(s', t)$, for all s' in the appropriate interval. Then by Lemma 6 we get that $\sigma_i(s') = t^*$ for all s' in this interval, contradicting Lemma 2. Therefore H_i must be differentiable. □

Lemma 9 (Differential equation system). *Every BE is such that for t in* $[0, T_i)$, $\varepsilon(\cdot)$ *and* $\zeta(\cdot)$ *as defined in Proposition* 1, *are a solution to* (1).

Proof. If $t = \sigma_j(z)$ is in $(0, \infty)$ for some z, by Lemma 6 it must solve the first-order condition $h_i(t) = -[1 - H_i(t)]z$. Let $\phi: [0, \infty) \to [i_L, i_H]$ and $\nu: [0, \infty) \to [j_L, j_H]$, where $\phi(t) = s$ if and only if $s = \sup\{v$ such that $\sigma_i(v) \le t\}$, and $\nu(t) = z$ if and only if $z = \sup\{u$ such that $\sigma_j(u) \le t\}$. Note that by Lemmas 2, 4 and 7, along the equilibrium path $\phi(t) = \varepsilon(t)$ and $\nu(t) = \zeta(t)$. Therefore the above condition can be rewritten as $h_i(t) = -[1 - H_i(t)]\zeta(t)$. Moreover, by Lemma 7,

$$H_i(t) = F_i(\varepsilon(t)) .$$

Differentiating, we get that

$$h_i(t) = f_i(\varepsilon(t))\varepsilon'(t).$$

Substituting into the first-order condition we get the desired equations. □

Lemma 10 characterizes the set of types that eventually concede.

Lemma 10 (Admission of defeat).

(i) *If* $i_H < 0$ *and* $j_H < 0$ *and the BE outcome is not an immediate agreement, then both players concede.*

(ii) *In every BE such that there is a positive probability that j will never give in,* $\sigma_i(s) < \infty$ *for all* $s_i < 0$.

Proof. By Lemma 9 strategies have to satisfy (1). Taking the first equation and dividing through, we obtain

$$\frac{-\zeta'(t)f_j(\zeta(t))}{[1-F_j(\zeta(t))]\varepsilon(t)}=1\,.$$

Since $\varepsilon(t)$ [as well as $\zeta(t)$] is strictly increasing, for any $t<t<T$,

$$\frac{-\zeta'(t)f_j(\zeta(t))}{[1-F_j(\zeta(t))]\varepsilon(T)}>1\,,$$

and moreover, integrating the inequality over the interval $[t, T]$, we obtain

$$-\frac{1}{\varepsilon(T)}\ln\left[\frac{1-F_j(\zeta(t))}{1-F_j(\zeta(T))}\right]>T-t\,.$$

Now note that as T tends to infinity, the right-hand side also does so, and therefore the left-hand side must do so too.

(i) Since $\varepsilon(T)<0$ by assumption, this means that $F_j(\zeta(T))$ must tend to 1 as T goes to infinity. The symmetric argument shows that Player i must also concede.

(ii) Since $1-F_j(\zeta(T))>0$ by assumption, this means that $\varepsilon(T)$ must tend to 0 as T goes to infinity. The symmetric argument shows that Player i must also concede. □

Finally, Lemma 11 proves existence and uniqueness if $i_H>0$ and $j_H>0$ (existence only if $i_H\le 0$ and $j_H\le 0$) of solutions to (1) that are compatible with the requirements of Lemmas 5 and 10.

In what follows we let

$$I=[i_L,\min\{0,i_H\})\times[j_L,\min\{0,j_H\})\,,$$
$$L=\{(\lambda,\mu)\in I,(\lambda-i_L)(\mu-j_L)=0\}\,,$$

and

$$H=\{(\eta,\varphi)\in[i_L,i_H]\times[j_L,j_H],\,(\eta-i_H)(\varphi-j_H)=0\}\,.$$

Also, Figs. 1 to 5 are useful.

Lemma 11 (Existence and (non)uniqueness). *For each pair of types $(\lambda,\mu)\in L$, there is a unique solution to* (1) *such that $\lim_{t\to 0}(\varepsilon(t),\zeta(t))=(\lambda,\mu)$. Moreover:*

(i) *If $0<i_H$ and $0<j_H$, there is a unique solution to* (1) *such that* $\lim_{t\to 0}(\varepsilon(t),\zeta(t))\in L$, *with* $\lim_{t\to T}(\varepsilon(t),\zeta(t))=(0,0)$ *for some $T\le\infty$. For this unique solution $T=\infty$.*

(ii) *If $i_H=0$* and *$j_H\le 0$, then all solutions to* (1) *such that* $\lim_{t\to 0}(\varepsilon(t),\zeta(t))\in L$, *with* $\lim_{t\to 0}\varepsilon(t)<i_H$ *or* $\lim_{t\to 0}\zeta(t)<j_H$, satisfy $\lim_{t\to T}(\varepsilon(t),\zeta(t))\in H$, *and $T=\infty$.*

(iii) *If $i_H<0$ and $j_H<0$, then all solutions to* (1) *such that* $\lim_{t\to 0}(\varepsilon(t),\zeta(t))\in L$, *with* $\lim_{t\to 0}\varepsilon(t)<i_H$ *or* $\lim_{t\to 0}\zeta(t)<j_H$, *satisfy* $\lim_{t\to T}(\varepsilon(t),\zeta(t))=(i_H,j_H)$ *for some $T\le\infty$.*

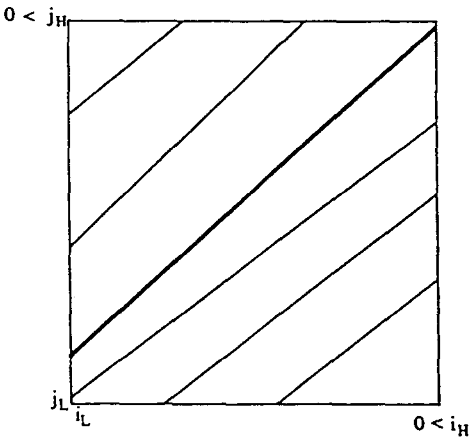

Fig. 1.

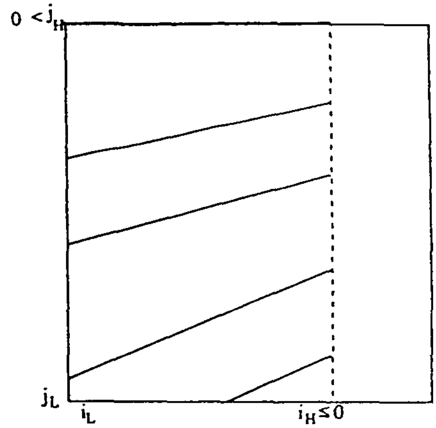

Fig. 2

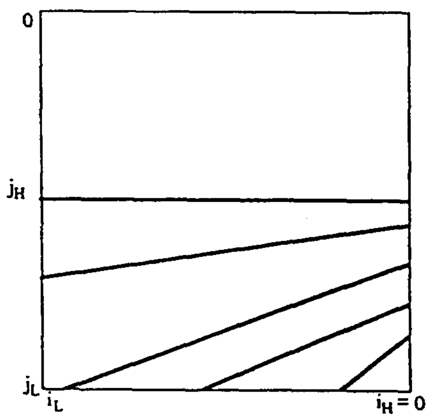

Fig. 3

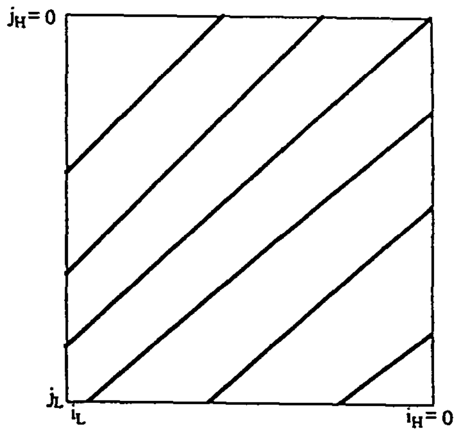

Fig. 4

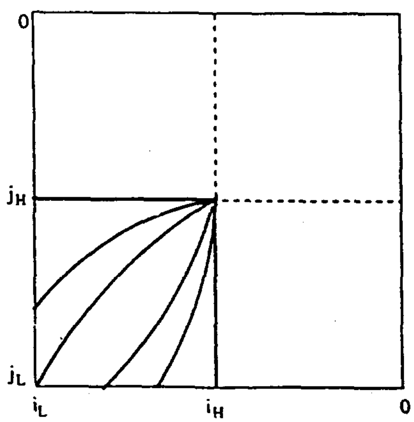

Fig. 5

—— Integral curves that yield an equilibrium

—— Integral curves that do not yield an equilibrium

Proof. By the Fundamental Existence–Uniqueness Theorem (see, for example, Davies and James, 1966, ch. 1) there is a unique solution to (1) such that $(\varepsilon(0), \zeta(0)) = (u, v)$ for all $(u, v) \in I$. This proves our first assertion.

For each $(u, v) \in I$, let $(\varepsilon_{uv}, \zeta_{uv})$ denote the solution to (1) such that $(\varepsilon(0), \zeta(0)) = (u, v)$. Each $(\varepsilon_{uv}, \zeta_{uv})$ is represented in I by its integral curve, i.e. the set $\{(s, z) \in I$ such that $s = \varepsilon_{uv}(t)$ and $z = \zeta_{uv}(t)$ for some $t \geq 0\}$. An integral curve of a solution to (1) is a solution to the differential equation

$$\frac{dz}{ds} = \frac{[1 - F_j(z)]sf_i(s)}{[1 - F_i(s)]zf_j(z)} . \tag{2}$$

For each $(u, v) \in I$, let $z_{uv}(\cdot)$ denote the unique solution to (2) such that $z(u) = v$. Note that at each point $(u, v) \in I$ the (unique) integral curve of a solution to (1) with initial condition $(\varepsilon(0), \zeta(0)) = (u, v)$ is fully characterized by z_{uv}.

(i) (See Figs. 1 and 2.) The nature (existence and uniqueness/multiplicity) of the solutions sufficiently near the singular point $(0, 0)$ can be explored using a first-order Taylor approximation (see Davies and James, 1966, ch. 2). That is, taking the Taylor expansions at $(0, 0)$,

$$\frac{[1 - F_j(z)]}{f_j(z)} s = \frac{[1 - F_j(0)]}{f_j(0)} s + o(2)$$

and

$$\frac{[1 - F_i(s)]}{f_i(s)} z = \frac{[1 - F_i(0)]}{f_i(0)} z + o(2) ,$$

in a neighborhood of $(0, 0)$ the nature of solutions to (2) can be determined from the solutions to

$$\frac{[1 - F_i(0)]}{f_i(0)} z \, dz = \frac{[1 - F_j(0)]}{f_j(0)} s \, ds . \tag{2'}$$

Let $\rho = [[1 - F_i(0)]f_j(0)/[1 - F_j(0)]f_i(0)]^{1/2}$, it is easily checked that $z + \rho s = 0$ and $z - \rho s = 0$ are the unique solutions to (2′) that go through $(0, 0)$. Only the second yields a curve in I, therefore there is only one solution to (1) such that $\lim_{t \to T}(\varepsilon(t), \zeta(t)) = (0, 0)$ for some $T \leq \infty$ and $\lim_{t \to 0}(\varepsilon(t), \zeta(t)) \in L$.

We now prove that $T = \infty$. Assume that there is a solution to (1), such that $\lim_{t \to 0}(\varepsilon(t), \zeta(t)) \in L$ and $\lim_{t \to T}(\varepsilon(t), \zeta(t)) = (0, 0)$ for some $T < \infty$. Note that $(\varepsilon(T), \zeta(T)) = (0, 0)$ and by (1) $(\varepsilon'(T), \zeta'(T)) = (0, 0)$. Differentiating again and again we get that all the derivatives of $(\varepsilon(\cdot), \zeta(\cdot))$ are $(0, 0)$ at T, and hence (by Taylor's Theorem) $(\varepsilon(\cdot), \zeta(\cdot))$ is locally constant at T. So there is some $T' < T$ such that $\lim_{t \to T'}(\varepsilon(t), \zeta(t)) = (0, 0)$, contradicting that a unique integral curve of (1) goes through $(0, 0)$.

(ii) (See Figs. 3 and 4.) All integral curves of (1) in I are increasing and do not cross, thus for each $(\lambda, \mu) \in L$ if $\lim_{s \to 0} z_{\lambda\mu}(s)$ exists it must be equal to some φ,

$j_L \leq \varphi \leq 0$. Analogously, if we represent integral curves as functions of z, if $\lim_{z \to j_H} s_{\lambda\mu}(z)$ exists it must be equal to some η, $i_L \leq \eta \leq 0$. This proves that $\lim_{t \to T}(\varepsilon(t), \zeta(t)) \in H$.

Next we check that for each $(\eta, \varphi) \in H$ there is a unique solution to (1) such that $\lim_{t \to T}(\varepsilon(t), \zeta(t)) = (\eta, \varphi)$ for some $T \leq \infty$. Without loss of generality we let $\eta = 0$. Taking Taylor expansions at $(0, \varphi)$,

$$\frac{[1 - F_j(z)]}{f_j(z)} s = \frac{[1 - F_j(\varphi)]}{f_j(\varphi)} s + o(2)$$

and

$$\frac{[1 - F_i(s)]}{f_i(s)} z = -\varphi s + o(2) ,$$

the nature of solutions to (2) in the neighborhood of $(0, \varphi)$ can be determined from the solutions to

$$\frac{\mathrm{d}z}{\mathrm{d}s} = -\frac{[1 - F_j(\varphi)]\varphi}{f_j(\varphi)} . \tag{2''}$$

Solutions to (2″) are of the general form

$$z = -\frac{[1 - F_j(\varphi)]\varphi}{f_j(\varphi)} s + K ,$$

and only $K = \varphi$ yields a solution such that $\lim_{s \to 0} z(s) = \varphi$.

Finally, that $T = \infty$ in the case $\lim_{t \to 0} \varepsilon(t) < i_H$ or $\lim_{t \to 0} \zeta(t) < j_H$, is proved as in (i).

(iii) (See Fig. 5.) Note that the unique solution to (2) such that $z(u) = j_H$ must be $z_{ui_H}(s) = i_H$ for all s. Analogously, the set $\{(u, v)$ such that $u = i_H$ and $0 < v < i_H\}$ is also an integral curve of some solution to (1). All integral curves of (1) are increasing and do not cross at any point in I. Therefore for any $(\lambda, \mu) \in L$, the unique solution to (2) such that $z(\lambda) = \mu$ must be such that $\lim_{s \to i_H} z_{\lambda\mu}(s) = j_H$. □

We are now ready to complete the proof of Proposition 1.

Proof of Proposition 1.

(i) By Lemmas 0 and 10 there is some $T < \infty$ such that i concedes no later than T with probability 1. If $0 < T$, then there is some δ, $T > \delta > 0$, such that j does not concede in $(T - \delta, T]$, since the conditional probability that i will concede in that interval is one. Then by Lemma 1 $H_i(t)$ is constant in $(T - \delta, T]$, contradicting the definition of T. Therefore, $T = 0$. On the other hand, since by Lemma 0 there is a positive probability that j does not give in at zero, then, by Lemma 5, j will not concede at 0.

(ii) Since along the equilibrium path ε and ϕ, and ζ and ν are equivalent, Lemmas 0, 1, 5 and 9 imply that any equilibrium outcome must be characterized by a solution to (1). By Lemmas 0, 3, 7 and 10, such a solution must be strictly increasing with $\lim_{t\to T_i} \varepsilon(t) = 0$ and $\lim_{t\to T_i} \zeta(t) = 0$. By Lemma 5 we must also have $\lim_{t\to 0}(\varepsilon(t)\zeta(t)) \in L$. By Lemma 11(i), (1) has a unique solution satisfying this condition, and $T = \infty$ in this solution.

We now check that the proposed strategies indeed constitute an equilibrium. First note that by Lemma 6 the objective function of every type is continuous. By Lemmas 4 and 8 the density of concessions is positive for all $t > 0$, and the objective function is differentiable in $(0, T)$ with the derivative at t equal to $h_j(t) + (1 - H_j(t))s$ for player i of type s. By (1), $h_j(t) = -\zeta'(t)f_j(\zeta(t)) = [1 - F_j(\zeta(t))]\varepsilon(t)$. Therefore, at each $t > 0$, we have $\zeta'(t)f_j(\zeta(t)) > (<) - [1 - F_j(\zeta(t))]s$, for all $s > (<)\varepsilon(t)$: That is, the objective function is strictly increasing (decreasing) at all dates t, $0 < t < \sigma_i(s)$ $(t > \sigma_i(s))$. Moreover, for types $s > \lim_{t\to 0} \varepsilon(t)$, the objective function is also strictly increasing at 0: if $H_j(0) > 0$, it is clear that the expected payoffs of player i are strictly increasing at $t = 0$. If $H_j(0) = 0$, since $\lim_{t\to 0}[h_j(t) + (1 - H_j(t))\varepsilon(t)] = 0$, we have that $\lim_{t\to 0}h_j(t) = \lim_{t\to 0}\varepsilon(t)$; for types $s \le \lim_{t\to 0} \varepsilon(t)$ the right derivative of the payoffs at $t = 0$ is $-\lim_{t\to 0} \varepsilon(0) + s < 0$. Therefore, for all s such that $\sigma_i(s) > 0$, the first-order condition must yield an optimum. All others will prefer to yield at 0 just as we claimed. Hence, the proposed strategy is indeed a BE.

(iii) If one of the players concedes at zero with probability 1 it is a best response never to concede. This supports the two immediate agreement BE. If there is a positive probability of later agreement, then by Lemma 9 the strategies have to satisfy (1). By Lemma 5 we need $\lim_{t\to 0}(\varepsilon(t)\ \zeta(t)) \in L$.

If $i_{\mathrm{H}} < 0$ and $j_{\mathrm{H}} < 0$, Lemma 10 requires that $\lim_{t\to T} \varepsilon(t) = i_{\mathrm{H}}$ and $\lim_{t\to T} \zeta(t) = j_{\mathrm{H}}$. By Lemma 11(iii) there is a continuum of such solutions. Optimality can be shown as in (ii).

Let $i_{\mathrm{H}} = 0$. By Lemma 10, either $\lim_{t\to T} \varepsilon(t) = 0$ and $\lim_{t\to T} \zeta(t) = \varphi \le j_{\mathrm{H}}$, or $\lim_{t\to T} \varepsilon(t) = \eta \le 0$ and $\lim_{t\to T} \zeta(t) = j_{\mathrm{H}}$. By Lemma 11(ii), any solution to (1) such that $\lim_{t\to 0}(\varepsilon(t)\zeta(t)) \in L$, with $\lim_{t\to 0} \varepsilon(t) < i_{\mathrm{H}}$ or $\lim_{t\to 0} \zeta(t) < j_{\mathrm{H}}$, satisfies $\lim_{t\to T}(\varepsilon(t)\zeta(t)) \in H$ and $T = \infty$. Optimality can be shown as in (ii) for all types that concede at some $t \le \infty$. The same argument also proves that types $z \in [\varphi, j_{\mathrm{H}}]$ cannot gain by conceding at some finite date either: payoffs are strictly increasing at 0 for them too and the derivative of their payoff is strictly positive at all $t > 0$. □

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References

D.T. Bishop, J. Cannings and J. Maynard Smith, The war of attrition with random rewards, J. Theoret. Biol. 74, (1978) 377–388.

C. Bliss and B. Nalebuff, Dragon-slaying and ballroom dancing: the private supply of a public good, J. Pub. Econom. 25 (1984) 1–12.

K. Chatterjee and W.F. Samuelson, Bargaining with two-sided incomplete information, Rev. Econom. Studies 54 (1987) 175–192.

T.V. Davies and E.M. James, Nonlinear Differential Equations (Addison-Wesley, 1966).

D. Fudenberg and J. Tirole, A theory of exit in Duopoly, Econometrica 54 (1986) 943–960.

D. Fudenberg, R. Gilbert, J. Stiglitz and J. Tirole, Preemption, leapfrogging and competition in patent races, Europ. Econom. Rev. 22 (1983) 3–31.

R. Ghemawat and B. Nalebuff, Exit, The Rand J. Econom. 16 (1985) 184–194.

K. Hendricks, A. Weiss and C. Wilson, The war of attrition in continuous time with complete information, Int. Econom. Rev. 29 (1988) 663–680.

D.M. Kreps, and R.B. Wilson, Reputation and imperfect information, J. Econom. Theory 27 (1982) 253–279.

J. Maynard Smith, The theory of games and the evolution of animal conflicts, J. Theoret. Biol. 47 (1974) 209–221.

P. Milgrom and R. Weber, Distributional strategies for games with incomplete information, Math. Op. Res. 10 (1985) 619–632.

B. Nalebuff and J. Riley, Asymmetric equilibria in the war of attrition, J. Theoret. Biol. 113 (1985) 517–527.

M. Osborne, The role of risk aversion in a simple bargaining model, in A.E. Roth, ed., Game Theoretic Models of Bargaining (Cambridge University Press, Cambridge, 1985), 181–213.

J.A. Ordover and A. Rubinstein, A sequential concession game with asymmetric information, Quart. J. Econom. 101 (1986) 879–888.

C. Ponsati, Unique equilibrium in a model of bargaining over many issues, Annales d'Economie et de Statistique 25/26 (1992) 81–100.

C. Ponsati and J. Sákovics, Mediation is necessary for efficient bargaining, UAB-IAE W.P. 194.92 Universitat Autònoma de Barcelona (1992).

C. Ponsati and J. Sákovics, Multi-person bargaining over two alternatives, Games Econ. Behav., forthcoming (1995).

J. Riley, Strong evolutionary equilibrium and the war of attrition, J. Theoret. Biol. 83 (1980) 383–402.