Implementation by decent mechanisms[*]

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Abstract

We address the design of optimal mechanisms for bargaining problems subject to incomplete information on the reservation shares of the agents. Decent rules are those that are Pareto Optimal in the constrained set of rules satisfying strategy proofness, individual rationality and weak efficiency - a requirement on *ex post* efficiency. We characterize decent rules and prove a uniqueness and existence theorem for a large class of utilities, including the constant relative risk aversion utilities. The decent rule can be close to *ex post* efficient when the agents are very risk averse. We prove that decent rules are implemented in type-monotone equilibrium of the Filtered Demands game, a continuous-time bargaining game, whereby under incomplete information players submit their claims over time to a passive agent who minimizes transmission of information between the players. In the Filtered Demands game the designer doesn't have to know either the distribution of the agents' types, nor their utilities.

KEYWORDS: Mechanism Design. Bargaining Problem. Incomplete Information. Dominant Strategies. Risk Aversion. Implementation.

1 Introduction

The contributions of this paper are illustrated by an example of bilateral trade. Consider a buyer and a seller that wish to exchange an indivisible object. Assume that their reservation values are private information, while their utilities over net gains are common knowledge.

Agents can be risk averse. It is well known that if agents act strategically and trade is voluntary, it is impossible to assure first best allocations. [1] We study trading mechanisms that satisfy *incentive compatibility*, *individual rationality*, and *weak efficiency*. Weak efficiency imposes that whenever the reservation price of the buyer exceeds that of the seller trade takes place with *ex post* positive probability.[2] We define a decent rule as a Pareto-optimal mechanism satisfying the three above properties. We then provide two sets of results. First, we prove existence and uniqueness of the decent rule, and show that it can be close to *ex post* efficient when the agents are very risk averse. Direct implementation of the decent rule requires that the designer and the agents know the utility forms of both agents. Second, we relax this common knowledge requirement by constructing an indirect dynamic implementation game form, which we call the *Filtered Demands (FD) game*. The FD game implements the decent rule for any concave utilities of the agents. While the utilities still have to be common knowledge among the agents the mechanism designer no longer needs to know that information.

We study a bargaining problem where two agents must share a unit of surplus. The agents have reservation shares, the fraction of the surplus below which they prefer no agreement. They have concave utilities over their personal net gain (that is after subtracting the reservation share). Utilities are common knowledge, but their reservation shares are private information. Thus, the restricted domain of preferences of each agent can be parametrized by her reservation share. Possible outcomes are defined as two-point lotteries between any feasible division and the "no agreement". A bargaining rule is a social choice function, mapping agents' reservation shares into outcomes. This formulation encompasses bilateral trade and the problem of sharing the cost of a public good. More precisely, the bilateral trade problem is as described above, and the cost-sharing of a public good problem is one where agents have private valuations of the public good and must split its cost.

We focus on bargaining rules that satisfy the strongest incentive requirements, *strategy proofness* (SP) - also called the *ex post incentive compatibility*, *ex post individual rationality* (IR), and a distinctive property which we call *weak efficiency* (WE). Weak efficiency requires

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[1] For dominat strategy incentive compatibility see Hurwicz [1972], Green and Laffont [1977], and D'Aspermont and Gerard-Varet (1979). For Bayesian environments see Myerson and Satterthwaite [1983].

[2] A non-cooperative game of incomplete information can be divided into three temporal stages. At the *ex-ante* stage each agent knows only the distributions of types of all agents, including himself. At the *interim* stage each agent knows own type but still knows only the distribution of types of her opponents. At the *ex post* stage the types of all agents are common knowledge.

that whenever agents' reservation shares are revealed to be compatible, the *ex post* probability that all the surplus is allocated is strictly positive.[3] A decent rule is a Pareto-optimal rule among the rules satisfying SP, IR and WE.

While SP and IR are standard conditions with clear economic intuitions, WE is a bit more intricate. One possible justification for WE is that it forces the mechanism to be responsive to agents' preferences. This implies that when agents' incentives are easily kept in line, we should expect a decent mechanism to be quite efficient even in the *ex ante* sense.[4] A more economically meaningful interpretation of WE comes from the following thought experiment. Suppose the two agents agreed *ex ante* to participate in a mechanism where they could always renegotiate an inefficient outcome with some pre-specified sequence of mechanisms. For example, if in the first round *mechanism 1* gave an inefficient outcome they would resort to *mechanism 2*, if that one produced an inefficient outcome they would renegotiate using *mechanism 3*, and so on. By using such a procedure, the probability of compatible types never agreeing would tend to zero, as the number of rounds of renegotiation would tend to infinity. In other words, any types that were compatible, would eventually agree. Furthermore, if the agents required this whole procedure to be strategy proof, then by the revelation principle there would exist an equivalent direct revelation mechanism. Interpreting the loss due to discounting as the probability of "no agreement", this direct mechanism would thus have to satisfy weak efficiency. The decent mechanism can thus be interpreted as the direct implementation in dominant strategies of an outcome of a dynamic game where agents are always able to renegotiate inefficient outcomes, and agree as soon as possible. While WE seems innocuous, it is in fact a very powerful condition in combination with the incentive constraints.

We fully characterize decent rules in environments where the agents' utilities are concave and the joint distribution of players' types satisfies the conditional independence, also called the spanning property. We prove the existence and uniquenes of the decent rule for a general class of concave utilities. The decent rule is always probabilistic, the probability of

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[3]Precise timing of a probabilistic mechanism in the *interim* and *ex post* stages is important. At the end of the *interim* stage the agents report their private parameters to the mechanism. The mechanism then computes the outcome - a lottery which is technically resolved immediately. There are two possible interpretations. The more appropriate one is that the agents only find out the final outcome of the lottery. The other interpretation is that at the beginning of the *ex-post* stage the agents observe the lottery itself. The lottery is then in fact resolved at the end of the *ex-post* stage.

[4]It is important to distinguish between the rules that do well in the *ex ante* sense, those that do well in the *interim*, and the ones that do well *ex post* - a rule that does well *ex ante*, might sometimes do very poorly *ex post*. In this sense, weak efficiency is a minimum requirement on *ex post* efficiency.

implementing the disagreement point being the tool to elicit truthful revelation. Moreover, its outcomes depend non-trivially on agents' reports, meaning that a decent mechanism is responsive to agents' preferences. Consequently, the probability of agreement among compatible types increases as the agents become more risk averse. In the limit, as agents become infinitely risk averse, the probability of trade approaches 1, or full efficiency, for all compatible types. The intuition behind this result is very clear: when the agents are very risk averse, it becomes "cheap" to provide them with the correct incentives for reporting truthfully, as long as the mechanism is responsive to their reports. For a particular class of utilities that includes all utilities with constant relative risk aversion (CRRA) we compute the decent rule in closed form: it is a *fixed-share rule* - one where compatible types receive their reported reservation value and split the revealed net surplus in constant shares.

Our work is related to Hagerty and Rogerson [1987]. They prove in the context of bilateral trade with risk-neutrality, that any strategy proof mechanism is *ex-ante* equivalent to some posted-price mechanism. In the bargaining context we call such mechanisms *posted-split* mechanisms, and they operate in the following way. First, a distribution function is announced, and the pie is split according to a random draw from that distribution. The split is then announced publicly, and if both agents agree to it, it is implemented. Otherwise the agents get their disagreement payoffs. It is easy to see that the most *ex-ante* efficient posted-split rule is always one where the distribution of the split is a degenerate one: for any given distribution of agents' types, the mechanism should pick the split that *ex-ante* maximizes the expected gains from trade. [5]

Because a decent mechanism is necessarily probabilistic, it could only be *ex-ante* equivalent to a posted-split rule with a non-degenerate distribution. Thus, when agents' utilities are quasi-linear, it is clear that the optimal posted-split rule will do *ex-ante* better than the decent rule. On the other hand, we show that when agents are sufficiently risk averse, the *ex-ante* optimal mechanism is no longer a posted-split mechanism, since a decent mechanism becomes more efficient. Another advantage of the decent rule is the following. To know the optimal posted-split rule, the designer has to know the distribution of agents' types. If this information is unknown to the designer, the efficiency of a posted split rule

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[5]The characterizations of strategy proof rules by Barberà and Jackson[1995] and Sprumont [1991] are also related. Barberà and Jackson [1995] show that in two-person economies strategy proofness and individual rationality imply that trade is at fixed proportions. For the bargaining problems with single peaked preferences, when the disagreement is not an alternative, Sprumont [1991] proves that the uniform rule is the unique rule that is efficient, symmetric and strategy proof.

will depend on luck alone, and such mechanism can be very inefficient. A decent mechanism, although perhaps not optimal *ex-ante*, will always be decent in terms of efficiency, even when the distribution of agents' types is not known. See Section 6 for a concrete example. Still, in order to compute the appropriate decent mechanism, the designer has to know the utility forms of the agents, whereas the incentive properties of the posted-split mechanisms are robust to this information.

The FD game implements the decent rule for all utility forms of the agents, as long as those are common knowledge among the agents, thus essentially decentralizing the mechanism. The FD game is a bargaining game in continuous time. In continuous time, impatient agents continuously keep sending their demanded shares of the pie to a central device, *the Filter*.[6] The Filter's role is to record these messages secretly, making them public only when the agents' demands become feasible. The information flow between the agents is thus minimized, since they can only observe the time that has passed until an agreement is reached. Then the game ends with the agents obtaining their agreed shares. In other words, the agents recognize how much net surplus is available only when they reach an agreement, and at that moment they share all the remaining surplus. In the FD game we define an M-equilibrium as a type-monotone undominated Bayesian equilibrium in regular strategies. Regular strategies satisfy some differentiability requirements. We prove that the M-equilibria of the FD game implement precisely the decent rules. The reason is that no common prior is needed in an M-equilibrium, implying that the implemented rule is strategy proof. [7] This implementation result is quite strong since, for any utility forms of the agents, there is a one-to-one correspondence between the M-equilibrium of the FD game and the decent rule.

The FD game also provides a strong link to the literature on non-cooperative bargaining games with incomplete information. It can be viewed as a relatively simple and natural extension of the Rubinstein-Stahl alternating-offers bargaining game[8] to continuous time in an incomplete information environment. The FD game provides a simple alternative to characterizing the set of equilibria of alternating-offers bargaining games with two-sided incomplete information. [9] Moreover, when bargaining is face-to-face, bargainers may be

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[6]Agents are assumed to have a common discount function, but this assumption is made mainly for the expositional convenience.

[7]The first order condition of the agents' best replies shows that the M-equilibria of the FD game are belief-independent, hence they are ex post equilibria. See for instance Ledyard [1978] or Bergemann and Morris [2003] .

[8]See Rubinstein [1982].

[9]Fully characterizing this set has so far remained an elusive task. See Ausubel, Cramton and Denekere

reluctant to make any relevant proposals in order not to disclose their private information and lose the strategic possibilities that private information gives. In other words, the agents may have incentives to reveal their private information very slowly, so that the scope for useful credible communication is severely limited, if not inexistent. [10] Interpreting the agents concession rates as "offers", our results demonstrate what can be attained when face to face bargaining is ruled out. Alternatively, the agents can be thought to be cognitively constrained so that they don't update their beliefs in response to opponent's offers. This means that bargainers learn only what their opponent cannot yield, so their beliefs change smoothly over time. Since the decent rule is the most efficient among the weakly efficient ones, our implementation results strongly suggest that in these settings completely restricting communication between the agents may be an efficient thing to do - learning is a double-edged sword. [11]

The idea of drastically filtering communication has previously been explored in Jarque, Ponsatí and Sákovics [2003]. They assume that concessions must take place in discrete steps so that the set of possible agreements is discrete. While this characteristic is often natural, the discretization of the set of partitions of the surplus comes at the expense of great technical problems. The set of Bayesian equilibria is very large; they all depend on the distribution of types, and so does their existence and efficiency performance. [12]

The rest of the paper is organized as follows. In Section 2 we formally describe the mechanism design problem. We define and characterize decent rules and we show their existence and uniqueness. In Section 3 we present the Filtered Demands game. In Section 4 we characterize its equilibria. In Section 5 we show that the equilibria of the FD game implement decent rules and that all decent rules are attained via the FD game. For environments of CRRA utilities we explicitly compute the equilibrium. In Section 6 we illustrate our results with an application to bilateral trade and provide some welfare comparisons. In Section 7

[2002] for an excellent survey and references.

[10] As a consequence, fully separating equilibria of alternating-offers bargaining game with two-sided incomplete information exhibit the undesirable property that, in the limit as the time interval between the offers vanishes, the probability of agreement vanishes too. See Theorem 1 in Ausubel and Denekere [1992].

[11] This justifies similar procedures that are used in practice. For example, the limit order book of the Paris Bourse allows hidden orders so as to increase the efficiency of the exchange. The FD game is an extreme example of such situation.

[12] If we drop the regularity requirement in the equilibria of the FD game, then the set of Bayesian equilibria also contains the equilibria which are similar to those in Jarque, Ponsatí, and Sakovics [2003]; if an agent believes that the opponent will only concede in discrete steps, then it only makes sense to concede at the complementary splits.

we conclude and discuss the extensions. We put most of the proofs in the Appendix.

2 Decent Bargaining Rules

Two agents, $i = 1, 2$, bargain over a unit of surplus. Denote by λ_i the share of the good that gets allocated to i. Index j will always indicate the agent other than i, i.e. $j \neq i$, for $i = 1, 2$. We will denote a vector (x_i, x_j) by x.

The *set of bargaining alternatives*, A, is the set of feasible divisions union the disagreement point. Formally, $A = \{\lambda | \lambda_1 + \lambda_2 \leq 1\} \cup d$. The preferences of agent i over A are represented by a utility function $u_i(\lambda_i, s_i)$, $u_i : A \times [0, 1] \to R$, where $s_i \in [0, 1]$ is agent's type. Agent's type s_i represents her reservation share, the share that leaves her indifferent to disagreement, and is her private information. The payoff from disagreement is normalized to 0, and thus $u_i(s_i, s_i) = u_i(d, s_i) = 0$. Agents' preferences over $\Delta(A)$, the set of lotteries over A, are represented by their expected utilities. Most of our analysis will be carried over under the simplifying assumption that (u_1, u_2) are of the form $u_i(\lambda_i, s_i) = u_i(\lambda_i - s_i)$ where $u_i(.) : [0, 1] \to R$ is a twice differentiable, strictly increasing and concave, with $u_i(0) = u_i(d - s_i) = 0$. Note that in this case $\frac{\partial u_i(\lambda_i - s_i)}{\partial \lambda_i} = u_i'(\lambda_i - s_i)$.[13] We also assume that $\lim_{x \to 0} \frac{x u_i'(x)}{u_i(x)} = K_i$, where $K_i \in (0, \infty)$. From now on we fix a pair $u = (u_1, u_2)$ and we pose a mechanism design problem *in the restricted domain of preferences* determined by u.

We define a *probabilistic bargaining rule* (Y, P) to be a direct revelation mechanism[14], mapping pairs of reports $z = (z_1, z_2)$ into two-point lotteries $P(z) \otimes Y(z) + (1 - P(z)) \otimes d$. Thus the rule (Y, P) prescribes disagreement with probability $(1 - P(z))$, and agreement at $Y(z) = (Y_1(z), Y_2(z))$ with probability $P(z)$, where $Y_i(z)$ is the share of the surplus allocated to agent i, $0 \leq Y_i(z) \leq 1$, and $Y_1(z) + Y_2(z) = 1$. Given (Y, P) the expected utility of agent i of type s_i upon reports z, $U_i(s_i, z)$, is

$$U_i(s_i, z) = u_i(Y_i(z) - s_i) P(z).$$

We consider the following properties of bargaining rules.

(SP) - STRATEGY PROOFNESS. (Y, P) is *strategy proof* if truthful reports constitute

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[13]We could generalize results to the more general class of differentiable utilities, $u_i(\lambda_i, s_i)$, satisfying the concavity and single-crossing conditions, formally stated in Appendix. This would complicate the exposition, without providing new intuitions, so we prefer to focus our analysis on the simpler case $u_i(\lambda_i - s_i)$.

[14]We will be dealing with the rules that are not manipulable in dominant strategies. Hence, we can appeal to the revelation principle for the dominant strategy environments and identify the set of non-manipulable social choice functions with the set of direct revelation mechanisms that implement them.

a dominant-strategy equilibrium. That is: $U_i(s_i, s) \geq U_i(s_i, z_i, s_j)$ for every $z_i \neq s_i$, all $s_i, s_j \in [0, 1]$, $i = 1, 2$.

(IR) - EX POST INDIVIDUAL RATIONALITY. (Y, P) is *ex post individually rational* if $P(s) > 0$ implies that $u_i(Y_i(s), s_i) \geq 0$ for all $s \in [0, 1] \times [0, 1]$, $i = 1, 2$.

(WE) - WEAK EFFICIENCY. (Y, P) satisfies *weak efficiency* if

$$s_1 + s_2 < 1 \Rightarrow P(s_1, s_2) > 0.$$

We are now ready for a formal defininition of decent rules.

(Q-D) - QUASY-DECENT RULES. A *quasi-decent bargaining rule* is one that satisfies SP, IR and WE.

(D) - DECENT RULES. A *decent bargaining rule* is a Pareto- optimal rule among the quasi-decent bargaining rules.

Before going on we introduce some additional notation. Since we are only interested in the rules that are individually rational, we can rewrite $Y_i(s) = s_i + \theta_i(s)\rho$,$i = 1, 2$, where $\theta_1(s) + \theta_2(s) = 1$, $0 \leq \theta_i(s) \leq 1$ and $\rho = 1 - s_1 - s_2$ denotes the revealed net surplus. Thus, θ_i represents the net share of the revealed surplus that goes to agent i. From now on, we will denote a bargaining rule either by (Y, P), or by (θ, P), where $\theta = \theta_1$ and $\theta_2 = 1 - \theta$, and θ is as defined above. It will also be convenient to define $w_i(x) = \frac{u_i'(x)}{u_i(x)}$. The first example shows that WE is not a trivial requirement.

Example 1. *Consider a rule which prescribes a split at predetermined and fixed shares, a posted-split rule, and the agents divide the surplus if they both agree to such division. Thus, $Y_1(z) = y_1$, $Y_2(z) = y_2 = 1 - y_1$, where $y_1 \in [0, 1]$, and y_1 is constant with respect to z. Clearly, such a rule satisfies SP and IR, but does not satisfy the WE.*

Since WE is an open ended constraint it could happen that decent mechanisms weren't well defined. However, it turns out that Q-D mechanisms form a one parametric family, which is parametrized by the efficiency of a given mechanism so that decency is a well defined concept. This is a trivial consequence of Theorem 5. In the following example we show that decent mechanisms exist. We also provide most of the economic intuition for what we do in the rest of the section. The main problem in finding a decent mechanism is to find a continuous rule for the division of the surplus, such that when an agent miss-reports her reservation share, she is either punished by a lower probability of implementing the solution, or by a lower surplus allocation. Proposition 6 generalizes the next example.

Example 2. *Assume that the utilities of the agents are linear: $u_i \left(\lambda_i - s_i \right) = \lambda_i - s_i$. Then a mechanism defined by $P \left(s \right) = \begin{cases} \pi \left(1 - s_1 - s_2 \right) & ; \ if \ s_1 + s_2 < 1 \\ 0 & ; \ otherwise \end{cases}$, $Y_i \left(s \right) = \frac{1}{2} \left(1 + s_i - s_j \right)$, $i = 1, 2$, is quasi-decent. To see that compute*

$$U_i \left(z_i, s_j; s_i \right) = \pi \left(1 - z_i - s_j \right) \frac{1}{2} \left(1 + z_i - s_j - s_i \right).$$

It is immediate to see that this quadratic function has a maximum precisely at $z_i = s_i$. Theorem 5 below will show that these are all the differentiable quasi-decent mechanisms for this case. Hence the unique decent one is obtained by setting $\pi = 1$.

Now we turn to characterizing decent mechanisms. It is no coincidence that the mechanism in Example 2 is continuous and monotonic in the sense that both $P(.)$ and $Y(.)$ are continuous and monotonic functions of the agents' reports. In the following lemma, we show that this is always the case for decent mechanisms.

Lemma 3. *(i) If a bargaining rule (Y, P) satisfies SP and IR then, for $i = 1, 2$, $Y_i \left(s_1, s_2 \right)$ is monotonically increasing in s_i, $P \left(s_1, s_2 \right)$ is monotonically decreasing in s_i, and Y_i and P are continuous at all $\left(s_1, s_2 \right)$ s.t. $u_i \left(Y_i \left(s_1, s_2 \right), s_i \right) > 0$. (ii) If a bargaining rule additionally satisfies WE then Y and P are continuous everywhere.*

Proof. See Appendix. $\square$

We now provide a first order condition for general strategy-proof mechanisms in the spirit of the Mirrlees approach to mechanism design. The condition is the standard local incentive constraint: if an agent miss-reports, her gains in a more favorable allocation have to be offset by a lower probability of implementing that allocation, and vice-versa. Since $P(.)$ and $Y_i(.)$ have to be by the previous lemma monotonic and continuous, they have to be piece-wise continuously differentiable. From now on, we assume that they are differentiable and that θ is differentiable as well.[15]

Proposition 4. *A bargaining rule (θ, P) is decent if and only if the following hold:*

1. If $s_1 + s_2 < 1$ then $P \left(s \right) > 0$ and θ_i and P satisfy

$$\frac{\frac{\partial P(s)}{\partial s_i}}{P \left(s \right)} = -w_i \left(\rho \theta_i \left(s \right) \right) \left(\theta_j \left(s \right) - \frac{\partial \theta_j \left(s \right)}{\partial s_i} \rho \right) ; \ i = 1, 2, \qquad (1)$$

for all $s \in [0, 1]^2$, with the additional condition $P \left(0, 0 \right) = 1$, where $w_i \left(x \right) = \frac{u_i'(x)}{u_i(x)}$.

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[15]See Copic [2003] for a deeper discussion of this issue. From the analysis there, it is aparent that the assumption here is in fact without loss of generality.

2. *If $s_1 + s_2 > 1$ then $P(s) = 0$.*

Proof. We give a proof of necessity, similar to the intuition of Example 2. Consider a decent rule (Y, P). By IR and WE there is no loss of generality in requiring that $Y_1(z) = z_1 + \theta(z)(1 - z_1 - z_2)$, and $Y_2(z) = z_2 + (1 - \theta(z))(1 - z_1 - z_2)$ where $0 \le \theta(z) \le 1$. The condition $P(s) = 0$ for all s such that $s_1 + s_2 > 1$ is necessary for individual rationality. Also, $0 < P(s) \le 1$ for $s_1 + s_2 < 0$ is necessary for weak efficiency.

Strategy proofness is equivalent to the requirement that for every s_i and z_j the function $U_i(z_i; s_i, z_j) = P(z_i, z_j) u_i(Y_i(z_i, z_j), s_i)$ has a global maximum at the point $z_i = s_i$. Since (Y, P) must be differentiable (a.e.) by Lemma 3, the necessary first order condition for a maximum is that

$$\frac{\partial U_i(s_i; s_i, z_j)}{\partial z_i} = 0 \text{ for all } s_i, z_j.$$

That is

$$\frac{\partial P(s_i, z_j)}{\partial z_i} u_i(Y_i(s_i, z_j) - s_i) + P(s_i, z_j) u_i'(Y_i(s_i, z_j) - s_i) \frac{\partial Y_i(s_i, z_j)}{\partial z_i} = 0$$

for all s_i, z_j, and $i = 1, 2$, $j \ne i$. It is clearly enough that these conditions hold for all possible truthful reports of agent j, i.e. when $z_j = s_j$ for all s_j. Substituting $\theta(s_1, s_2)$ in $Y_i(s_i, s_j)$ yields (1).

To see that $P(0, 0) = 1$ is necessary assume that $P(0, 0) = \pi < 1$ and that $P(., .)$ is differentiable everywhere. The monotonicity of the mechanism implies that $P(s_1, s_2) < \pi < 1$ for all (s_1, s_2). Take an alternative rule (Y', P') where for all (s_1, s_2) the sharing rules are the same $Y_i'(s_1, s_2) = Y_i(s_1, s_2)$ and the probabilities of agreement $P'(s_1, s_2) = \frac{P(s_1, s_2)}{\pi}$ are increased. It is immediate to check that the rule (Y', P') still satisfies (1). Moreover, it is strictly preferred by all types, contradicting decency of (P, Y).

For the proof of sufficiency (i.e. that local incentive constraint is enough to guarantee the correct incentives globally) see Appendix. $\square$

Remark 1. *Take a decent rule (θ, P). As we noted above, multiplying P by a constant $\pi \in (0, 1)$ gives us a quasi-decent rule. Also notice that any quasi-decent rule is invariant with respect to multiplying utilities by constants. That is, if (P, Y) is a quasi-decent rule for utilities (u_1, u_2) then it is also a quasi-decent rule for $(C_1 u_1, C_2 u_1)$ where $C_i > 0, i = 1, 2$.*

In the rest of this section we provide uniqueness and existence theorems for decent rules, and we characterize the decent rules for a relatively simple class of utilities.

Theorem 5. *Whenever a decent rule exists, it is unique. Moreover if (Y, P) is the decent rule for a given vector of utilities, then all differentiable quasi-decent rules are given by $(Y, \pi P)$, $\pi \in (0, 1]$.*

Proof. See Appendix. $\square$

We already saw in Example 2 that the decent mechanism exists when utilities are linear. Does it exist for any other class of utilities? The simplest rules we could hope for are the ones that are linear in types, meaning that the shares of revealed suplus are constant. We call these fixed share rules.

(FS) - FIXED SHARE RULES. A *fixed-share rule* is a rule for which $\theta_1(s_1, s_2) = \theta^*$, and $\theta_2(s_1, s_2) = 1 - \theta^*$, for some constant θ^*, $0 \le \theta^* \le 1$.

Slightly more general are the rules that only depend on the revealed surplus.

(NS) - NET SURPLUS RULES. A *net-surplus rule* is a rule (Y^{ns}, P^{ns}) where the probability of agreement and the share of the net surplus assigned to each agent depend only on the net surplus. That is,

$$
\begin{aligned}
Y_i^{ns}(s) &= s_i + \theta_i(\rho)\rho, i = 1, 2, \\
P^{ns}(s) &= \begin{cases} P(\rho), & \rho > 0 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

In the next proposition we fully characterize the class of utilities for which decent fixed-share mechanisms exist.

Proposition 6. *Assume that for $i = 1, 2$, utility of agent i is of the form $u_i(\lambda_i - s_i)$. Then a quasi-decent fixed share mechanism exists if and only if either of the following holds:*

1. *$u_1(x) = Cu_2(x)$, $C > 0$, for every $x \in [0, 1]$. In this case the unique decent rule is a net surplus mechanism, given by*

$$
\begin{aligned}
\theta_1 &= \theta_2 = \theta^* = \frac{1}{2} \\
P^*(\rho) &= \frac{u_1\left(\frac{1}{2}\rho\right)}{u_1\left(\frac{1}{2}\right)}
\end{aligned}
$$

2. *For $i = 1, 2$, agent i has a utility of the form[16]:*

$$
u_i(\lambda_i - s_i) = C_i(\lambda_i - s_i)^{\gamma_i} e^{D_i(\lambda_i - s_i)}, C_i > 0,
$$

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[16]It is easy to see that u_i is increasing and concave on $[0, 1]$ if and only if $\gamma_i \in (0, 1]$ and $D_i \in \left[-\gamma_i - \sqrt{\gamma_i}, \sqrt{\gamma_i} - \gamma_i\right]$. Notice also that when $D_i = 0$ this utility function is $u_i(\lambda_i - s_i) = (\lambda_i - s_i)^{\gamma_i}$ which is the constant relative risk aversion (CRRA) utility. Also note that in this case, the mechanism is a net surplus mechanism.

$$\gamma_i \in (0,1], \ D_i \in [-\gamma_i - \sqrt{\gamma_i}, \sqrt{\gamma_i} - \gamma_i]$$

In this case a unique decent rule is a fixed-share mechanism, given by

$$\theta^* = \frac{\gamma_1}{\gamma_1 + \sqrt{\gamma_2 \gamma_1}}$$

$$P^*(\rho) = \begin{cases} e^{-D_1 \theta_1 s_1 - D_2 \theta_2 s_2} \rho^{\sqrt{\gamma_1 \gamma_2}}; \ s_1 + s_2 \leq 1 \\ 0; \ otherwise. \end{cases}$$

Proof. See Appendix. $\square$

To round up this Section, we provide the existence theorem for the decent rule for the general case.

Theorem 7. *If the utilities of the agents are of the form $u_i (\lambda_i - s_i)$, then the decent mechanism exists and is a net surplus mechanism. It is characterized by the following differential equations for P and θ:*

$$\rho \theta'(\rho) + \theta(\rho) - \frac{w_1 (\rho \theta(\rho))}{w_2 (\rho (1 - \theta(\rho))) + w_1 (\rho \theta(\rho))} = 0,$$

with initial condition at 0, $\theta(0) = \frac{K_1}{K_1 + \sqrt{K_1 K_2}}$, where $w_i(x) = \frac{u_i'(x)}{u_i(x)}$, $K_i = \lim_{x \to 0} x \, w_i(x)$, $i = 1, 2$, and

$$\frac{P'(\rho)}{P(\rho)} = w_2 (\rho (1 - \theta(\rho))) (\theta(\rho) + \rho \theta'(\rho))$$

with initial condition at 1, $P(1) = 1$.

Proof. See Appendix. $\square$

We have thus proven the existence of the decent rule. In general, we can't analytically solve the above differential equations, especially the equation for θ. One possibility is to assume the form of θ, and then compute the utilities for which such θ is the decent rule. On the other hand, we can in general numerically compute P and θ. The reason that the numerical computation is non-problematic is that at the boundary $\rho = 0$, where the equation (1) becomes explosive, we know what values the functions P and θ have to take. So any numerical method will be stable, if we start at that boundary.

3 The Filtered Demands Game

In this section we propose a dynamic bargaining game implementing decent bargaining rules.

THE GAME. The *Filtered Demands Game* (FD game) is a continuous-time game. The agents send private messages, claiming shares of the good, to the Filter. The Filter is a dummy player whose only role is to receive claims, keeping them secret while they are incompatible, and to announce the agreement as soon as it is reached. As time goes by, the agents can continuously decrease their demands at any moment. Thus the agents revise their claims until they become mutually compatible. Then the Filter announces that agreement has been reached, the agents receive the agreed shares, and the game ends.

STRATEGIES. A *strategy $\lambda_i(.,.)$ of player i* is a function mapping her type s_i and time t into a share,

$$\lambda_i(.,.) : [0,1] \times [0,\infty) \to [0,1] \ , \ i = 1,2.$$

Thus $\lambda_i(s_i,t)$ is the share agent i of type s_i claims for herself at $t \geq 0$. Strictly speaking, a strategy is a function mapping each type and each history into a proposal at every moment. However, given her type s_i, the history at time t only depends on t, as the agent is not able to see the proposals of her opponent.

Because the time is continuous, we have to be careful in order to have well-defined outcomes. For that we need condition (FD1). It can be thought of either as a restriction on admissible actions at each $t \in R_+$ through the rules of the game, or as a behavioral assumption. The agents are only allowed to increase their pledges to the other player, and at each moment they have to set the rate of change of their pledge in a continuous way. [17]

FD1 $\lambda_i(s_i,t)$ is differentiable w.r.t. t, with $\frac{\partial \lambda_i(s_i,t)}{\partial t} \leq 0$ for all $t \in [0,\infty)$, $s_i \in [0,1]$.

OUTCOMES. If two strategies are such that $\lambda_1(s_1,0) + \lambda_2(s_2,0) < 1$, i.e. the demands are more than compatible at $t = 0$, then agreement between types (s_1,s_2) occurs at $t = 0$ at shares $\lambda_i(s_i,0) + \frac{1-\lambda_1(s_1,0)-\lambda_2(s_2,0)}{2}$. [18] Given a pair of types s, a strategy profile λ determines

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[17]For a discussion on admissible strategies and sensible outcomes in continuous-time games see Simon and Stinchcombe[1989]. In order to have well-defined outcomes, we could weaken condition (FD1) to left continuity w.r.t. time. Then, the agents might in a sequentially rational way believe that the opponent will almost surely bid only finitely many intermediate agreement points between the two extreme agreements. Best response to such a strategy is to bid only the complementary intermediate agreement points (since any other bid is essentially irrelevant). Such strategies would give rise to equilibria like those described by Jarque et. al [2003]. Differentiability w.r.t. t clearly eliminates this type of equilibria.

[18]Our results are independent of the excess sharing rule, as long as it gives positive shares to both agents.

a unique outcome of the game denoted by $(x_1(\lambda, s), x_2(\lambda, s), \tau(\lambda, s))$, where x_1 and x_2 are the shares of agents 1 and 2, and $\tau(\lambda, s)$ is the first time of agreement, that is

$$\tau(\lambda, s) = \min\{t | \lambda_1(s_1, t) + \lambda_2(s_2, t) \leq 1\}.$$

This minimum is well defined because strategies are continuous.

For a given type s_i of player i and time t we can also specify types of player j that agree with s_i at time t. Let $\tilde{s}_j(.,.;\lambda) : [0,1] \times R_+ \twoheadrightarrow 2^{[0,1]}$, where $s_j \in \tilde{s}_j(s_i, t; \lambda)$ iff $t = \tau(\lambda, s_i, s_j)$. In other words, $\tilde{s}_j(s_i, t; \lambda)$ is a correspondence that maps types s_i of player i and times t into types of player j who enter in agreement with s_i at t. From now on, we suppress the λ in the arguments of $\tilde{s}_j$. Finally, we define *the entry time $t_i^E(s_i)$* as the first time when s_i could agree with some type of player j. That is, $t_i^E(s_i) = \min\{t | \tilde{s}_j(s_i, t) \neq \emptyset\}$.

INTER-TEMPORAL UTILITIES. The static utility of agent i is given by the function $u_i(\lambda_i, s_i)$, satisfying the same requirements as in Section 2. Again, we assume for simplicity that $u_i(\lambda_i, s_i) = u_i(\lambda_i - s_i)$. Agents discount the future exponentially.[19] Thus upon agreement at $t \geq 0$ at a share λ_i, the payoff of agent i is given by

$$U_i(\lambda_i, s_i, t) = e^{-t} u_i(\lambda_i - s_i).$$

Note that in the event of perpetual disagreement the payoffs are zero.

INFORMATION AND BELIEFS. It is common knowledge that pairs of reservation values s are drawn from a continuous joint distribution function F, with a positive density f on $[0,1] \times [0,1]$.[20] Agent i knows her own type, and has beliefs over the distribution of the opponent's types. As noted above, the histories depend only on t. Thus, given a strategy profile λ, the beliefs of a player about the opponent are updated only as a function of time. We denote by $F_j(s_j | s_i, t; \lambda)$ the distribution of the belief of agent i of type s_i about agent j at time t, conditional on no agreement until time t. By $f_j(s_j | s_i, t; \lambda)$ we denote the density of F_j, whenever it exists. Finally, we denote by $H_j(s_i, t; \lambda)$ the mass of types of player j with whom agent i has agreed with by time t. We will economize the notation and omit parrameter s_i and λ whenever that is unambiguous. Note that if the strategies of both

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[19]This is for convenience and can be realxed to any discounting criterion $\delta(t)$, where $\delta(.)$ is a strictly positive, monotonically decreasing function with $\delta(0) = 1$, and $\lim_{t \to \infty} \delta(t) = 0$. See Proposition 16.

[20]We could generalize our analysis to any F with positive density f on a square $[\underline{s}, \overline{s}] \times [\underline{s}, \overline{s}]$, $\underline{s} < 1/2 < \overline{s}$. This is equivalent to the requirement that F has support on $[\underline{s}, \overline{s}] \times [\underline{s}, \overline{s}]$, and the conditional beliefs of agents are independent. In the literature, this condition also appears as the "spanning condition" (see for instance Mookherjee and Reichelstein[1992], p395).

players are differentiable with respect to both parameters, and these partial derivatives are non-zero, the beliefs will be differentiable with respect to time.

EQUILIBRIUM. We are interested in undominated type-monotonic Perfect Bayesian Equilibria (PBE) of the FD game. In the subsequent paragraphs we will make precise what undominated means in our setup.

Before we define the PBE, note two more facts. First, in our setup, the set of PBE outcomes is equal to the set of Bayesian Equilibrium (BE) outcomes. The reason is that off-equilibrium histories either end the game or are unobservable to the opponent, hence own deviation from a BE cannot be optimal at any t. Second, the PBE concept only makes sense for the case of exponential discounting. [21]

Let $EU_i\left(s_i; \lambda, F\right)$ denote the expected payoff of player i of type s_i, when agents play according to strategy profile λ and types are distributed according to F. Let $F_j\left(s_i\right)$ denote the conditional distribution of j's types. Thus,

$$EU_i\left(s_i; \lambda, F\right) = \int_0^1 u_i\left(x_i\left(\lambda, s_i, s_j\right) - s_i\right) e^{-\tau\left(\lambda, s_i, s_j\right)} dF_j\left(s_i\right),$$

or alternatively

$$EU_i\left(s_i; \lambda, F\right) = \int_{t\in[0,\infty)} u_i\left(\lambda_i\left(s_i, t\right) - s_i\right) e^{-t} dH_j\left(s_i, t\right),$$

where both of these integrals have to be understood as Lebesgue integrals.

Denote by Λ_i the set of strategies for player i, that is, Λ_i is the set of functions $\lambda_i\left(., .\right)$ satisfying (FD1). A strategy profile λ constitutes a *Bayesian equilibrium* if and only if

$$EU_i\left(s_i; \lambda, F\right) \geq EU_i\left(s_i; \bar{\lambda}_i, \lambda_j, F\right), \forall\bar{\lambda}_i \in \Lambda_i,$$

for all $s_i \in [0, 1]$, $i = 1, 2$, $j \neq i$.

A careful definition of the PBE in our setting requires specifying agents' expected utility in every subgame, which in our setup means at every time t. Let $EU_i\left(s_i, t; \lambda, F\right)$ denote the expected payoff to player i of type s_i in the subgame starting at t, when agents play strategies λ (this also specifies the history):

$$EU_i\left(s_i, t; \lambda, F\right) = \int_{\eta\in[t,\infty)} u_i\left(\lambda_i\left(s_i, \eta\right) - s_i\right) e^{-\eta} dH_j\left(s_i, \eta\right)$$

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[21]The PBE in a continuous time game imposes a temporal consistency on the agents' behavior and beliefs, and a necessary condition for this to hold is that the discounting be stationary, thus exponential.

A strategy profile λ constitutes a *Subgame Perfect Bayesian equilibrium* if

$$EU_i\left(s_i, t; \lambda, F\right) \geq EU_i\left(s_i, t; \bar{\lambda}_i, \lambda_j, F\right), \forall \bar{\lambda}_i \in \Lambda_i \text{ s.t. } \bar{\lambda}_i\left(s_i, \eta\right) = \lambda_i\left(s_i, \eta\right) \text{ for all } \eta \leq t,$$

for all $s_i \in [0, 1]$, for all $t \geq 0$, $i = 1, 2$, $j \neq i$.

As we noted earlier, BE and PBE are outcome-equivalent in our setup.

For each BE profile λ, a profile λ' constructed by adding a *stand still interval* $[0, T)$, i.e. $\lambda'_i\left(s_i, t + T\right) = \lambda_i\left(s_i, t\right)$, is a BE as well, for any $T < \infty$. As the opponent does not concede any positive amount until T, no concession prior to T is useful. Regardless of T, such strategy profiles λ' are weakly dominated.[22] We say that a BE is *undominated* if it does not have a stand still interval.

Strict type-monotonicity of strategies is a sufficient condition for an equilibrium to be fully separating. However, in order to prove the weak type-monotonicity and the existence of strictly type-monotone equilibrium, we have to further restrict the strategies to satisfy the following regularity condition.

FD2 We say that a strategy is *regular* if $\frac{\partial \lambda_i(s_i, t)}{\partial s_i}$ is continuous for all $t \in [0, \infty)$, and $\lim_{t \to \infty} \lambda_i\left(s_i, t\right)$ is a left-continuous function of s_i, for all $s_i \in [0, 1]$.

The first part of condition (FD2) imposes smoothness with respect to types. The second part is roughly an indifference breaking rule: if an agent of some type is at the horizon indifferent between two concessions to the opponent, she will concede more (see also the footnote in the proof of Lemma 9 in the Apendix). This condition is enough to assure that the continuity of the demands with respect to types is preserved at the time horizon. It is also enough to assure that at the time horizon the agents bid their types, in any BE. Since types and dates take values in a continuum, and the range of strategies is also a continuum, we can also think of the regularity condition as a natural pattern of behavior that rules out dramatic changes when types change only marginally.

Before focusing on the separating equilibria of the FD game, we prove some useful lemmas that have to hold in any BE. The first obvious observation is that agents prefer disagreement to negative payoffs at every moment.

Lemma 8. Ex Post Individual Rationality: *In any BE, $\lambda_i(s_i, t) \geq s_i$ for all t and all s_i.*

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[22]In the next section we will show that the type of player i who makes the earliest relevant offer is $s_i = 0$. Type $s_i = 0$ has nothing to lose if she starts moving at 0, since she has no reason to expect some other type to start moving any earlier. This, in turn, would provoke other types to start moving as well.

In the next lemma we state that claimed shares asymptotically approach the reservation values. The intuition is that if an agent of a given type doesn't reach agreement in a very long time, the opponent is probably of a high type, so that the agent should lower her demand. She would only keep lowering it until her type[23].

Lemma 9. (DELAYED) EX POST EFFICIENCY: *In any regular PBE,* $\lim_{t \to \infty} \lambda_i (s_i, t) = s_i$ *for all* $s_i \in [0, 1]$.

Proof. See Appendix. □

We next assert that agents with high reservation values, "tougher" agents, never demand less than "softer" ones, whenever their demands matter. The meaning of the condition in the lemma below is that whenever nothing happens with probability 1 to a given type of a player, it doesn't matter what her demand is.

Lemma 10. WEAK TYPE MONOTONICITY: *In any regular BE,* $\frac{\partial \lambda_i (s_i, t)}{\partial s_i} \geq 0$, *for all times* $t \in (0, \infty)$ *and types* $s_i \in [0, 1]$, *which satisfy the condition that* $H_j (s_i, t)$ *is strictly increasing at* t.

Proof. See Appendix. □

After these general observations about the form of regular PBE, we from now on focus on the separating regular equilibria. To avoid any confusion, we give it a special name: *M equilibrium*, where M stands for monotone in types, regular and undominated.[24]

ME) - M EQUILIIBRIUM An M equilibrium of the FD game is a strictly type-monotone PBE in undominated and regular strategies.

4   Equilibria in the FD Game

In this section we characterize M equilibrium of the FD game. We will proceed as follows. First we will derive the first order condition (FOC) of agent's best reply, when her opponent uses a strictly type-monotone strategy. Using this FOC, we will then show that in equilibrium, a best reply to a strictly type-monotone strategy is strictly type monotone. Finally, in

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[23]This argument is only valid if the reservation demand of the "toughest type" is very high - that is if $s_H \geq 1$.

[24]We conjecture that there are no other regular BE outcomes in the FD game, but we haven't been able to prove that claim.

the next section we will show that there is a one-to-one correspondence between the set of M equilibria of the FD game and the set of decent rules. Hence, appealing to the uniqueness theorem for the decent rules, the M equilibrium of the FD game is unique as well.

First we discuss the optimization problem of an agent when her opponent uses a strategy that is regular and strictly increasing in types. After a lemma deriving the initial conditions for the agents' optimal strategies, we state the dynamic optimization program that the agents are facing. In the main proposition of this section we derive the first order condition, which turns out to be belief independent. We first focus on the initial conditions for the agents' strategies.

From Lemma 10, it follows that any s_i starts participating in the negotiations once her demand becomes feasible with the demand of $s_j = 0$. Before that moment the agent must know that she is demanding too much to agree even with the lowest type of the opponent. Therefore the question is: should an agent enter in the game already at $t = 0$ (and with what demand), or should she wait until the field softens up a bit. We denote by $g_i(s_i)$ the starting point of the demand of type s_i: $g_i(s_i) = \lim_{t \searrow 0} \lambda_i(s_i, t)$.

Lemma 11. INITIAL CONDITION: *In any M equilibrium it must hold that $g_1(0) + g_2(0) = 1$.*

Proof. In an M equilibrium the type $s_i = 0$ at time 0 demands a share that will give her a positive probability of agreement in at least a very short time - otherwise each type of every agent would know that there was some dead delay at the start where the only thing that would happen would be that agents would lower their demands up to the point where the lowest types could agree, violating the fact that M equilibrium is undominated. On the other hand, it cannot be that agent i demands a share which meets a demand of some type $s_j^0 > 0$ of player j - meaning that $g_i(0) + g_j(s_j^0) = 1$. This follows from the excess profit sharing rule since then the type $s_i = 0$ could profitably deviate by starting with a demand that met type $s_j = 0$. Then she would "rip off" all the excess agreement profits by lowering her demand very rapidly to $1 - g_j(s_j^0)$. By making her move fast enough it is clear that such deviation could be profitable. $\qquad \square$

Thus for all types except the lowest type it is in equilibrium optimal to wait with a high demand for a while. It means that there will necessarily be delays with probability 1.

Recall that we defined the entry time $t_i^E(s_i)$ as the first moment when agent i of type s_i makes a realistic proposal. It is a simple corollary to the above lemma that $t_i^E(s_i)$

is the moment when the demand of type s_i is compatible exactly with the lowest type of the opponent. The proof is exactly the same as the proof of Lemma 11. The remark that follows is equally simple.

Corollary 12. TIMING OF ENTRY: *In equilibrium, $\lambda_i\left(s_i, t_i^E\left(s_i\right)\right) = 1 - \lambda_j\left(s^L, t_i^E\left(s_i\right)\right)$, for $i = 1, 2$, $j \neq i$, and all $s_i \in [0, 1]$.*

Remark 2. *In any M equilibrium $t_i^E\left(s_i\right) < \infty$ if and only if $s_i < 1$. Otherwise the strategy of s_i would be strictly dominated.*

We are now ready to write down the dynamic optimization problem. In equilibrium, agents select a strategy aiming at the highest possible payoff, given the type-contingent strategies of the other player. Thus agents are picking optimal functions $\lambda_i(s_i, \cdot)$, $i = 1, 2$. This means that agent i of type s_i decides how her concessions of the good to the other side should optimally change with time. An important step in the proof of the Proposition 13 is to show that for any $(s_i, t) \in [0, 1] \times [t_i^E\left(s_i\right), \infty)$, $\tilde{s}_j\left(s_i, t\right)$ is a function (and not a correspondence), defined by

$$1 = \lambda_j(\tilde{s}_j\left(s_i, t\right), t) + \lambda_i(s_i, t). \tag{2}$$

This is a consequence of the assumption that the oponent plays a strictly type-monotone strategy, and the implicit function theorem.

Proposition 13. OPTIMIZATION PROGRAM: *If the strategy of agent j is regular and strictly type-monotone, then the best reply of agent i of type s_i solves the following optimization program*

$$Max_{\lambda_i(s_i, \cdot) \in \Lambda_i} \int_{[t_i^E(s_i), \infty)} e^{-t} u_i(\lambda_i(s_i, t) - s_i) f_j(\tilde{s}_j\left(s_i, t\right)) \frac{\partial \tilde{s}_j\left(s_i, t\right)}{\partial t} dt,$$

s.t. (2) and $t_i^E\left(s_i\right)$ is defined by the condition $\tilde{s}_j\left(s_i, t_i^E\left(s_i\right)\right) = 0$.

Proof. Fix the type of agent i to be s_i. When entering into negotiations at $t_i^E\left(s_i\right)$, she decides her optimal concession plan $\lambda_i\left(s_i, t\right)$, $t > t_i^E\left(s_i\right)$, in order to maximize her expected discounted future payoff. As in the proof of Lemma 10 denote by $H_j\left(t\right)$ the probability of type s_i reaching agreement up to time t (again, we omit the parameter s_i in $H_j\left(t; s_i\right)$). Agent i is solving the following program

$$Max_{\lambda_i(s_i, \cdot) \in \Lambda_i} \int_{[t_i^E(s_i), \infty)} e^{-t} u_i(\lambda_i(s_i, t) - s_i) dH_j\left(t\right)$$

19

But the possibility of reaching an agreement at some $t > t_i^E(s_i)$ is exactly the possibility that agent i will at t meet the demand of some type of agent j. For any $t \geq t_i^E(s_i)$, recall that $\tilde{s}_j(s_i, t)$ is the type of agent j with whom i reaches agreement at moment t. Thus $\tilde{s}_j(s_i, t)$ is implicitly defined from the relation

$$\lambda_j(\tilde{s}_j(s_i, t), t) + \lambda_i(s_i, t) = 1.$$

By definition and Lemma 11, [25] $\tilde{s}_j(s_i, t_i^E(s_i)) = 0$, and by Lemma 9 $\lim_{t \to \infty} \tilde{s}_j(s_i, t) = 1 - s_i$. Taking the derivative with respect to t, we can express

$$\frac{\partial \tilde{s}_j(s_i, t)}{\partial t} = -\frac{\frac{\partial \lambda_j(\tilde{s}_j(s_i, t), t)}{\partial t} + \frac{\partial \lambda_i(s_i, t)}{\partial t}}{\frac{\partial \lambda_j(\tilde{s}_j(s_i, t), t)}{\partial s_j}}.$$

By assumption, $\frac{\partial \lambda_i}{\partial t}$ and $\frac{\partial \lambda_j}{\partial t}$ are both finite and non-positive, and $\frac{\partial \lambda_j}{\partial s_j}$ is strictly positive. Hence, we can see from the implicit function theorem, that for any $t \geq t_i^E(s_i)$, $\tilde{s}_j(s_i, t)$ is a well defined differentiable function of time, with $0 \leq \frac{\partial \tilde{s}_j(s_i, t)}{\partial t} < \infty$. In other words, at any $t \geq t_i^E(s_i)$ there exists exactly one type $\tilde{s}_j(s_i, t)$ of player j, with whom s_i would reach agreement at that moment. These facts have two consequences. First, the probability of reaching an agreement by t, $H_j(t)$, has no mass points because the distribution of types of player j has no mass points. Second, the marginal increase in $H_j(t)$, $dH_j(t)$, is equal to the marginal increase of the mass of types of player j, that player i would agree with by moment t. Also, agent i knows that before $t_i^E(s_i)$ her proposals were unrealistic, so she cannot update her beliefs until that moment. Since $\tilde{s}_j$ is differentiable with respect to time, the beliefs are updated continuously and differentiably from $t_i^E(s_i)$ on. In other words, we have established that at $t_i^E(s_i)$ the belief of agent i is exactly $F_j(s_j)$, and at every moment $dH_j(t) = dF_j(\tilde{s}_j(s_i, t)) = f_j(\tilde{s}_j(s_i, t)) \frac{\partial \tilde{s}_j(s_i, t)}{\partial t} dt$. This completes the proof. $\qquad\square$

The optimization problem stated in Proposition 13 can be best approached as a problem where i is choosing two unknown functions $\lambda_i(s_i, \cdot)$ and $\tilde{s}_j(s_i, .)$ which are bound by the constraint (2), where $\lambda_j(\cdot, \cdot)$ is a given and fixed function (the strategies of all possible types of agent j). [26] The optimality condition at the lower boundary of optimization is given by definition of $t_i^E(s_i)$ - implicitly written it is $\tilde{s}_j(s_i, t_i^E(s_i)) = 0$. In the following lemma we provide the first order condition of the optimization program of agent i, for $t > t_i^E(s_i)$. To save space, we are omitting most of the arguments in the functions.

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[25]Corollary 12 also implies that at every instant there will be only one type reaching an agreement with any particular type of the other agent.

[26]Our manual for the calculus of variations is Elsgolts [1970].

Lemma 14. FIRST ORDER CONDITION: *In any M equilibrium the function $\lambda_i(s_i, .)$, $i = 1, 2$, satisfies the following first order condition*

$$u_i(\lambda_i - s_i) = u_i'(\lambda_i - s_i) \left(\frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial s_j} \frac{\partial \tilde{s}_j}{\partial t} + \frac{\partial \lambda_i}{\partial t} \right), \tag{3}$$

for every $s_i \in [0, 1]$ and every $t > t_i^E(s_i)$.

Proof. See Appendix. $\qquad\square$

Corollary 15. *To prove that any pair of strategies of agents 1 and 2 satisfying Equation (3) will indeed constitute an M equilibrium of FD game, we have to check that a strategy satisfying (3) will be monotone in agent's type. It is immediate to check this. Assume the contrary, that $\frac{\partial \tilde{s}_j}{\partial t} = 0$. Then, by assumption $\frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial s_j} > 0$, $\frac{u_i(\lambda_i - s_i)}{u_i'(\lambda_i - s_i)} > 0$, $\frac{\partial \lambda_i}{\partial t} \leq 0$, so if $\frac{\partial \tilde{s}_j}{\partial t} = 0$ the equation (3) can't be satisfied.*

Lemma 14 yields a condition that is independent of the beliefs of player i about the types of player j. This remarkable property is of importance for our results, and might seem surprising at first sight. It has, however, a natural intuitive explanation. Take as given the strategy of player j, a function λ_j that is strictly increasing and differentiable with respect to her type. Knowing the exact trajectory of j's demands over time *for any given type* s_j, player i of type s_i must simply decide when to propose a share just compatible with the demand $\lambda_j(s_j, t)$. Since the equilibrium is fully revealing, once the agreement occurs, the equilibrium must be an *ex post* equilibrium. That is, fixing an equilibrium, at any agreement agent i knows exactly what j's type is, and it is known *ex ante* that she will know. So i must be playing a best response to every type of player j, thus the first order condition has to be independent of beliefs.

For the sake of completeness we show that if agents' discounting functions are not exponential, the analysis changes only slightly[27]. Intuitively, the units in which time is measured, don't matter.

Proposition 16. GENERAL DISCOUNTING FUNCTIONS *Suppose that discounting is given by a general function $\delta(t)$ where $\delta(.)$ is a strictly positive, monotonically decreasing function with $\delta(0) = 1$, and $\lim_{t \to \infty} \delta(t) = 0$. Denote the equilibrium strategies for exponential discounting by $\bar{\lambda}_i(s_i, \tau)$. Then the equilibrium strategies for discounting $\delta(t)$ are given by $\lambda_i(s_i, t) = \min\{1, \bar{\lambda}_i(s_i, -\ln(\delta(t)))\}$.*

Proof. See Appendix. $\qquad\square$

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[27] This independence with respect to discounting could prove useful when designing experiments.

5 Existence of Equilibria and Implementation

IMPLEMENTED BARGAINING RULE. At $\tau(s)$ the agents get the shares $\lambda_i(s_i, \tau(s))$ and $\lambda_j(s_j, \tau(s))$. We can interpret $e^{-\tau(s)}$ as the probability of implementation of the prescribed shares. Hence, we will say that λ *implements* (Y, P) *if and only if the outcome associated to* λ *is such that* $Y_i(s) = \lambda_i(s_i, \tau(s))$, *and* $P(s) = e^{-\tau(s)}$.

Since time plays a crucial role in the present setup, a natural interpretation is that rather than selecting outcomes stochastically, bargaining rules allocate agreements over time. We may thus view the FD game as a *dynamic bargaining rule (* and its associated *dynamic direct revelation mechanism*[28]), (X, τ), where individuals report their type at $t = 0$ and are instructed to implement an agreement with shares $X_i(s)$ only at date $\tau(s) \in [0, \infty]$. If $\tau(s) = \infty$, then the prescribed outcome is disagreement.

The FD game in equilibrium implements decent bargaining rules. We will show that this is always true, independently of agents' utilities. Clearly, the question is whether any equilibria of FD game exist at all (the agents' strategy sets are non-compact). We will prove that there is a one-to-one correspondence between the set of equilibria of the FD game and the set of decent rules. Hence, for any utilities, an equilibrium of the FD game exist if and only if a decent mechanism exists. By the uniqueness theorem for decent mechanisms, we know that the equilibrium of the FD game will always be unique. All of this is summarized in the following lemmas and propositions.

Lemma 17. INDIVIDUAL RATIONALITY: *Equilibria of the FD game implement rules satisfying IR.*

Proof. This is a direct consequence of Lemma 8. □

Lemma 18. WEAK EFFICIENCY*: Equilibria of the FD game implement rules satisfying weak efficiency.*

Proof. Lemmas 9 and 10 imply that all pairs that produce a positive net surplus reach agreement at a finite date, which translates into WE. □

Lemma 19. STRATEGY PROOFNESS: *Equilibria of the FD game implement rules satisfying SP.*

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[28]*Sequential* revelation mechanism, discussed in Cramton [1985] and Ausubel and Denekere [1993] are a related notion. As our game, being dynamic is not of sequential moves, we find the adjective dynamic more accurate for our purposes.

Proof. By Lemma 14 equilibria of the FD game are belief independent. This implies that the implemented rule must be strategy proof (see for instance Ledyard[1978], Bergemann and Morris[2003]). □

Lemma 11 then implies that the FD game implements precisely decent rules. As we show in the next proposition, there is a one-to-one correspondence between the set of the equilibria of the FD game and the set of decent mechanisms.

Theorem 20. IMPLEMENTATION: *An equilibrium of the FD game implements a decent bargaining rule. Conversely, any decent bargaining rule is implementable as an equilibrium of the FD game.*

Proof. The previous three lemmas show that the rule implemented in the equilibrium of the FD game must be quasi-decent. Decency is then implied by the fact that equilibria of the FD game are by definition undominated, hence the probability of the Pareto-efficient outcome is always set to be maximal. For the converse, see Appendix. □

Corollary 21. FD-UNIQUENESS: *Whenever the equilibrium in the FD game exists, it is unique.*

Proof. This is a direct consequence of Theorem 5 and the previous proposition. □

These are the central results of this section. Proposition 4 shows how to *calculate* all decent rules, given one-parametric utility functions. Proposition 20 goes much further: regardless of the utilities, an equilibrium of the FD game implements a decent bargaining rule, as long as the two agents know each other's utilities. The designer does not need to know this information.

Unless the set of equilibria of the FD game is empty, this game must implement a decent bargaining rule. Computing an equilibrium requires solving self-referential equations (2) and (3). For the environments where we can compute the decent rule in closed form, it is by the Proposition 20 easy to compute the equilibria of the FD game. In the next proposition we provide the strategies of the agents if they both have CRRA utilities. A similar exercise can be repeated for the other cases where the decent rule has a closed form solution.

Proposition 22. CRRA *If agents have CRRA utilities, then the following type-contingent strategies are the unique equilibrium and they implement the θ^*-Fixed Share rule*[29]:

$$\lambda_i(s_i, t) = \min\left\{1, s_i + \theta_i^* e^{-\frac{t}{\sqrt{\gamma_1 \gamma_2}}}\right\}, i = 1, 2,$$

where

$$\theta_1^* = \frac{\gamma_1}{\gamma_1 + \sqrt{\gamma_2 \gamma_1}} \tag{4}$$

and $\theta_2^* = 1 - \theta_1^*$.

Proof. For a direct proof that these strategies satisfy the FOC of the FD game see Appendix. The rest follows from Propositions 6 and 20, and the previous corollary. □

6 Bilateral Trade and Welfare Comparison

In this section we provide an example of how the FD game and the decent mechanisms are related, and three examples that illustrate the relative *ex ante* performance of the decent rule in comparison to alternative mechanisms. In order to facilitate comparison with the relevant literature (Hagerty and Rogerson [1987] and Myerson and Satterthwaite[1983]) we re-cast our model into the bilateral trade framework. Player 1, the seller, can produce the good at a cost s_1, player 2, the buyer, values the good at $b_2 = 1 - s_2$.

We first illustrate the connection between the decent mechanism and the FD game. Suppose the agents have CRRA utilities over their net gains, given by $u_i(x) = x^{\gamma_i}$. Then the decent mechanism is given by (θ^*, P), where $\theta_1^* = \frac{\gamma_1}{\gamma_1 + \sqrt{\gamma_1 \gamma_2}}$ and $P(s_1, b_2) = (b_2 - s_1)^{\sqrt{\gamma_1 \gamma_2}}$. Thus, the price at which they trade is $p = s_1 + \theta^*(b_2 - s_1)$. This corresponds to the following equilibrium strategies in the FD game. The equilibrium price proposed by the seller is

$$\lambda_1(s_1, t) = min\left\{1, s_1 + \theta_1^* e^{-\frac{t}{\sqrt{\gamma_1 \gamma_2}}}\right\},$$

and the equilibrium price proposed by the buyer is

$$\lambda_2(b_2, t) = max\left\{1, b_2 - \theta_2^* e^{-\frac{t}{\sqrt{\gamma_1 \gamma_2}}}\right\}.$$

Now we turn to the welfare analysis. Explicit computations in terms of *ex ante* expected welfare depend on the utility functions, the distribution of reservation values, and

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[29]Notice that at the time when $1 = s_i + u_i^{-1}(u_i(\theta^*)e^{-t})$ this violates our assumption on differentiability of strategies, but the strategies are still differentiable a.e. Moreover, at that point, the demand of agent of type s_i is irrelevant, hence we can modify it slightly to make it smooth.

on how much information the designer has. The requirements of SP and WE imply that in the decent rule the inefficiency gets distributed "uniformly" across all types. A consequence is that the decent rule may be *ex ante* sub-optimal. The worst case for the decent rule is when agents are risk neutral. However, there are at least two instances when the decent rule is more efficient than the posted prices. First, when the agents are risk averse this fact is exploited in the decent rule while the posted prices are completely non-reactive to utility information. Second, the efficiency of the decent rule turns out to be robust to the information about the distribution of agents' types, which is not the case for the posted prices. In such cases the decent rule may out-perform the posted prices.[30]

Example 23. *Our first benchmark example is a situation where agents are risk neutral, the distribution of agents' types is uniform on $[0,1] \times [0,1]$, and the mechanism designer knows that. For this environment, IR rules maximizing ex ante welfare subject to either Bayesian incentive compatibility or SP are well known. By Myerson and Satterthwaite [1983] the optimal mechanism under Bayesian incentive compatibility is one that prescribes that trade takes place if and only if $b_2 \geq s_1 + \frac{1}{4}$. On the other hand, under SP by Hagerty and Rogerson [1987], any rule is ex ante equivalent to a posted-price mechanism. It is then immediate to check that voluntary trade at a deterministic fixed price $\frac{1}{2}$ maximizes expected ex ante total gains. We can thus compare the welfare attained under each of these two rules to what is attainable under the decent rule. It is immediate to check that the unique decent rule prescribes that trade take place with probability $P(s_1, b_1) = \max\{(b_2 - s_1), 0\}$, at a price $p(s_1, b_1) = \frac{b_2 + s_1}{2}$.*

1. *The unconstrained potential welfare is $W^u = \int_0^1 \int_{s_1}^1 (b_2 - s_1)\, db_2 ds_1 = \frac{1}{6}$.*

2. *The expected gains under the optimal Bayesian incentive compatibile rule are*

$$W^{Bic} = \int_0^1 \int_{s_1 + 1/4}^1 (b_2 - s_1)\, db_2 ds_1 = \frac{9}{64}.$$

Hence, approximately .85 of the potential gains W^U are attained.

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[30]A natural question that arises when the agents are risk averse is what is the *ex ante* most efficient strategy-proof rule. It turns out that it is either the optimal posted-price mechanism or the decent mechanism. For the proof of this see Copic [2003].

3. The expected gains under the optimal posted-price rule are

$$W^{pp} = \int\limits_{0}^{\frac{1}{2}} \int\limits_{0}^{\frac{1}{2}} (b_2 - s_1)\, db_2 ds_1 = \frac{1}{8},$$

which is .75 of W^U.

4. Finally, the expected gains under the decent rule are

$$W^d = \int\limits_{0}^{1} \int\limits_{s_1}^{1} (b_2 - s_2)^2\, db_2 ds_1 = \frac{1}{12},$$

which is $\frac{1}{2}$ of W^U.

If the agents are risk averse then the situation changes. Next example explicitly shows that as the agents become very risk averse, the decent mechanism converges toward full efficiency. In the next example we measure the social welfare as the sum of the utilities of both agents, but the results are largely independent of the measure, as long as it satisfies some mild requirements- for example, it is enough if it is bounded.

Example 24. *Assume both agents have utilities displaying constant relative risk aversion, and that their types are still uniformly and independently distributed. So the utility of the seller is $u_1(s_1, p) = (p - s_1)^{\gamma_1}$ and the utility of the buyer is $u_2(b_2, p) = (b_2 - p)^{\gamma_2}$. Then the decent mechanism is given by (θ^*, P), where $\theta_1^* = \frac{\gamma_1}{\gamma_1 + \sqrt{\gamma_1 \gamma_2}}$ and $P(s_1, b_2) = (b_2 - s_1)^{\sqrt{\gamma_1 \gamma_2}}$. When $\gamma_1 \gamma_2 \to 0$, that is when at least one of the agents becomes infinitely risk averse, $P(s_1, b_2) \to 1$ for all $b_2 > s_1$. Thus, the mechanism converges to full efficiency when one of the agents becomes infinitely risk averse. Notice also that the more risk-averse agent gets a lower share of the surplus. In other words, a risk-averse agent is willing to give up some surplus in order to decrease the incentives to miss-report by the risk-neutral agent, thus assuring a less risky lottery.* [31]

Now suppose that the agents' risk-aversion parameters are the same, $\gamma_1 = \gamma_2 = \gamma$, and compare the expected gains from trade under the decent mechanism and the optimal posted-price mechanism. Again, we first compute the total unrestricted gains as a function

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[31] One can show an analogous result in the FD game if both agents are equally risk averse but one is more impatient than the other. Thus, there is a big resemblance between an agent being risk averse and impatient. This interplay is worth exploring in future work.

of γ.

$$W^U(\gamma) = \int\limits_0^1 \int\limits_0^{b_2} \left(\left(\frac{b_2 - s_1}{2} \right)^\gamma + \left(\frac{b_2 - s_1}{2} \right)^\gamma \right) ds_1 db_2 = \frac{2}{2^\gamma} \frac{1}{(\gamma + 1)(\gamma + 2)}.$$

The expected gains under the decent mechanism are now

$$W^d(\gamma) = \int\limits_0^1 \int\limits_0^{b_2} (b_2 - s_1)^\gamma \left(\left(\frac{b_2 - s_1}{2} \right)^\gamma + \left(\frac{b_2 - s_1}{2} \right)^\gamma \right) ds_1 db_2 = \frac{2}{2^\gamma} \frac{1}{(2\gamma + 1)(2\gamma + 2)}.$$

Finally, the expected gains under the optimal posted-price mechanism are

$$W^U(\gamma) = \int\limits_0^1 \int\limits_0^{b_2} \left(\left(\frac{1}{2} - s_1 \right)^\gamma + \left(b_2 - \frac{1}{2} \right)^\gamma \right) ds_1 db_2 = \left(\frac{1}{2} \right)^{\gamma+1} \frac{1}{(\gamma + 1)}.$$

Clearly, as $\gamma \to 0$, the expected gains under the decent rule converge to the unrestricted gains from trade, while the gains under the optimal posted-price mechanism converge to one half of the unrestricted gains from trade. One can compute that in this case the decent mechanism becomes more efficient than the posted price at $\gamma = \frac{1}{2}$.

The asymptotic full efficiency of the decent rule as agents become infinitely risk averse is in sharp contrast with Example 23. Indeed, for the environment of Example 23, it is immediate to check that the decent rule is *ex ante* equivalent to the following indirect mechanism: a price is selected from a uniform distribution with support in $[0, 1]$ and trade occurs if and only if it is individually rational for both agents to trade at the realized price. Hence, the decent mechanism is surely not equivalent to the optimal posted-price mechanism. The reason is that the optimal posted-price mechanism is a degenerate price distribution, for any distribution of types that satisfies the full-spanning condition. It is also apparent that when agents become increasingly risk averse, the optimal posted-price mechanism (given the distribution of agents' types) remains inefficient since it is completely unresponsive to agents' preferences. In other words, when agents are sufficiently risk averse, the decent mechanism will exploit that information, while the posted prices don't.

Another instance when the decent rule may be a more efficient incentive-compatible solution is when the designer is not precisely informed about the distribution of types, even if the agents are risk-neutral. The next example shows that since the decent mechanisms are robust with respect to the designer's information on types' distribution while posted prices aren't, the loss of efficiency under the 'wrong' posted price rule may be great relative to the sub-optimality of the decent rule.

Example 25. *Assume that both agents are risk neutral. Suppose that the designer had no knowledge about the distribution of agents' reservation shares and that by the principle of insufficient reason he assumed that it was symmetric for both agents. He would then use the above posted-price rule, the optimal posted-price rule for such distribution. However, assume that the distributions of agents' reservation shares were in fact asymmetric. In particular, take an $\varepsilon > 0$, and assume that the reservation values of the agents were independent and their densities had the following forms:*

$$f_1(s_1) = \begin{cases} \varepsilon \ for \ 0 \leq s_1 \leq \frac{1}{2} \\ \left(\frac{1}{\varepsilon} - 1 + \varepsilon\right) \ for \ \frac{1}{2} < s_1 \leq \frac{1}{2} + \varepsilon \\ \varepsilon \ for \ \frac{1}{2} + \varepsilon \leq s_1 \leq 1 \end{cases}$$

$$f_2(b_2) = \begin{cases} \left(\frac{1}{\varepsilon} - 1 + \varepsilon\right) \ for \ 1 \geq b_2 \geq 1 - \varepsilon \\ \varepsilon \ for \ 1 - \varepsilon > b_2 \geq 0 \end{cases}$$

Some tedious, straight-forward calculus shows that then the gains from trade under the posted-price rule are equal to $\frac{1}{8}\varepsilon(3 - 4\varepsilon + 2\varepsilon^2)$ which converges to 0 as $\varepsilon \to 0$. On the other hand, the gains from trade under the decent rule are $\frac{1}{24}(6 - 27\varepsilon + 60\varepsilon^2 - 53\varepsilon^3 + 12\varepsilon^4 + 4\varepsilon^5) \to \frac{1}{4}$, as $\varepsilon \to 0$. In other words, for ε small enough, the decent rule extracts a sizeable portion of the possible gains from trade, whereas the posted-price rule extracts arbitrarily close to none.

7 Conclusion and Extensions

We have addressed the design of mechanisms for the bargaining problem where the agents' reservation shares are private information. For the environments with concave utilities, we have fully characterized bargaining rules that we call decent - those that are Pareto Optimal in the constrained set of rules satisfying individual rationality, weak efficiency, and strategy proofness. We have proved that when it exists, the decent rule is unique; we have proved the existence for a large set of utilities. We have proposed a simple dynamic market game, the FD game, which always implements the decent rule, regardless of the agents' utilities and discounting criterion. This implementation result is due to the fact that the equilibria of the FD game do not depend on agents' beliefs. The game protocol itself is simple. Neither a dictatorial principal designing complex contracts, nor strong commitments to assure the agents' obedience over time are required. The dynamic game thus provides a link between the weak efficiency and the renegotiation-proofness. It also provides a sharp prediction to the situations of bilateral bargaining under incomplete information when the agents' behavior is regular and their updating constrained.

The present work can be extended to address situations with more than two agents. In Copic and Ponsati [2003a] we address multilateral bargaining with private reservation shares. This generalization is appropriate to address the problems of when to supply, and how to share the cost of a public good when there are many agents. [32] In this case, the Filter can be envisioned as a central agent administering a public account. Individuals pledge their contributions towards the cost of the public good, and can increase their pledge at any time. The Filter assures that contributions are not publicly disclosed until the necessary amount has been pledged. Payments are made only if and when the project is carried out. In Copic and Ponsati [2003b] we discuss markets with many participants. There we generalize the FD game to a dynamic double auction with many sellers and buyers with private valuations of the object. In that case the FD game can be imagined as a market with continuous trading and a closed limit order book.

Our characterization of the decent rules can be interpreted in the spirit of the classical axiomatic approach to bargaining. Taking as the starting point a bargaining problem in which the disagreement point is not common knowledge, we characterize the rule that induces agents to reveal their private information, and assigns with a positive probability a Pareto-optimal solution to the revealed problem. Given that we require strategy proofness, this probability has to non-trivially depend on the agents' private information. It is important to note that the Pareto-optimal allocation of the decent rule does not in general coincide with the Nash bargaining solution. The reason for this is that while the decent rule satisfies all the other axioms of the Nash bargaining solution, SP and the Independence of Irrelevant Alternatives are conflicting requirements. For more on this see Čopič[2003].

Appendix

CONDITIONS ON UTILITIES:

U1 For every $s_i \in [0,1]$, $u_i(.,s_i)$ is a strictly increasing and concave function of λ_i, i.e. $\frac{\partial u_i(\lambda_i,s_i)}{\partial \lambda_i} > 0$ and $\frac{\partial^2 u_i(\lambda_i,s_i)}{\partial \lambda_i^2} \leq 0$.

U2 For every $\lambda_i \in [0,1]$, $u_i(\lambda_i,.)$ is strictly decreasing in s_i, i.e. $\frac{\partial u_i(\lambda_i,s_i)}{\partial s_i} < 0$. Also, $\frac{\partial^2 u_i(\lambda_i,s_i)}{\partial \lambda_i \partial s_i} \geq 0$.

U3 For every $s_i \in [0,1]$ $u_i(s_i,s_i) = u_i(d,s_i) = 0$.

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[32]Mailath and Postlewaite [1990] address the question of Bayesian incentive-compatible mechanisms for the environments with risk-neutral agents.

U4 For every $s_i \in [0, 1]$ $\lim_{\lambda_i \to s_i} \frac{\lambda_i \frac{\partial u_i(\lambda_i, s_i)}{\partial \lambda_i}}{u_i(\lambda_i, s_i)} = K_i$, where $0 < K_i < \infty$.

The requirement U1 is a standard concavity condition. The conditions U2 and U3 assure that s_i behaves as a reservation share, and U4 is a technical condition on the slope at 0.

PROOF OF LEMMA 3:

Proof. i) Assume (Y, P) satisfies IR and SP. We first prove monotonicity. By strategy proofness for all z_i, z_i' and z_j

$$P(z_i, z_j) u_i (Y_i (z_i, z_j) - z_i) \geq P(z_i', z_j) u_i (Y_i (z_i', z_j) - z_i)$$

and

$$P(z_i', z_j) u_i (Y_i (z_i', z_j) - z_i') \geq P(z_i, z_j) u_i (Y_i (z_i, z_j) - z_i')$$

Since u_i is strictly increasing these inequalities imply that for all $z_i > z_i'$, $P(z_i', z_j) \geq P(z_i, z_j)$ and $Y_i (z_i, z_j) \geq Y_i (z_i', z_j)$, so that (Y, P) must be monotone.

Continuity: IR and SP imply that Y_i and P must be continuous at all (s_1, s_2) such that $u_i (Y_i (s_1, s_2) - s_i) > 0$, for $i = 1, 2$.

We first show that if $Y_i (z_1, z_2) \geq z_1$, and Y_i is continuous at all (z_1, z_2) such that $u_i (Y_i (z_1, z_2) - z_i) > 0$, then so must be P, and vice-versa. Assume by way of contradiction that Y_i is continuous and P is discontinuous at some (z_1, z_2). Then there is an $\varepsilon > 0$ such that for all $\delta > 0$, there is a $z_1' \in (z_1 - \delta, z_1 + \delta)$ such that $|P(z_1', z_2) - P(z_1, z_2)| > \varepsilon$. Assume wlog that $P(z_1', z_2) \geq P(z_1, z_2) + \varepsilon$. Then for δ small enough, an agent of type z_1 must be better off reporting z_1' instead of her true type z_1: the continuity of Y_i implies that the possible loss in the allocated share is negligible, while a strictly positive gain in probability of agreement is attained. Proving that continuity of P implies continuity of Y_i is analogous.

Assume that both Y and P are discontinuous at some (z_1, z_2), s.t. $u_i (Y_i (z_1, z_2) - z_i) > 0$, and again wlog let $P(z_1', z_2) \geq P(z_1, z_2) + \varepsilon$. To assure that z_1 reports truthfully under such discontinuous P, the discontinuity in Y_i must be such that $Y_1 (z_1, z_2) \geq Y_1 (z_1', z_2) + \gamma$ for some $\gamma > 0$ in order to assure that agent one report truthfully (note that it is possible to find such γ by $u_i (Y_i (z_1, z_2) - z_i) > 0$ and IR). Since $Y_2 (z_1, z_2) = 1 - Y_1 (z_1, z_2)$ the discontinuity of P at (z_1, z_2) is such that z_2 cannot prefer to report truthfully when facing z_1, contradicting strategy proofness.

(ii) If (Y, P) also satisfies weak efficiency then it is continuous everywhere. This follows from (i), since WE and strategy proofness imply continuity at points (z_1, z_2) s.t. $u_i (Y_i (z_1, z_2) - z_i) = 0$. To see it, observe first that by monotonicity, Y_i has to be continuous in the neighbourhood of the line $z_1 + z_2 = 1$. By IR, this line is the boundary of the set of points (z_1, z_2) s.t. $u_i (Y_i (z_1, z_2) - z_i) = 0$. If P were discontinuous anywhere on the line, then by same arguments as in (i) the types at the points of discontinuity could gain by mis-representing. $\qquad\square$

PROOF OF SUFFICIENCY OF PROPOSITION 4:

Proof. By the Lemma 3 quasi-decent rules are continuous and monotonic, hence they are differentiable almost everywhere.

Consider $U_1 (z_1; s_1, z_2)$. It is enough to show that for all $z_1 > s_1$ the derivative of $U_1 (z_1; s_1, z_2)$ w.r.t. z_1 is decreasing whenever $U_1 (z_1; s_1, z_1) > 0$ (deviations that give negative expected utility cannot be profitable). Thus compute

$$\frac{dU_1 (z_1; s_1, z_2)}{dz_1} = \frac{\partial P (z_1, z_2)}{\partial z_1} u_1' (Y_1 (z_1, z_2) - s_1) +$$

$$+ P (z_1, z_2) u_1' (Y_1 (z_1, z_2) - s_1) \left[1 + \frac{\partial \theta (z_1, z_2)}{\partial z_1} (1 - z_1 - z_2) - \theta (z_1, z_2) \right]$$

From the first order condition we can express

$$\left(1 + \frac{\partial \theta}{\partial s_1} (1 - z_1 - z_2) - \theta (z_1, z_2) \right) = - \frac{u_1 (Y_1 (z) - z_1)}{u_1' (Y_1 (z) - z_1)} \frac{\frac{\partial P(z)}{\partial s_1}}{P (z)}$$

Substituting this into the previous expression we get

$$\frac{dU_1 (z_1; s_1, z_2)}{dz_1} = \frac{\partial P (z)}{\partial z_1} \left[u_1 (Y_1 (z) - s_1) - \frac{u_1' (Y_1 (z) - s_1) u_1 (Y_1 (z) - z_1)}{u_1' (Y_1 (z) - z_1)} \right]$$

From here we see that whenever $u_1 (Y_1 (z_1, z_2) - s_1) > 0$, $\frac{dU_1(z_1; s_1, z_2)}{dz_1}$ is a decreasing function of z_1. This follows directly from the fact that u_1 is increasing and concave in λ_1. Thus the local maximum of U_1 is unique, and is also a global maximum. Similarly for U_2. $\qquad\square$

PROOF OF THEOREM 5:

Proof. Recall the SP constraint for decent mechanism:

$$\frac{\partial P}{\partial s_1} u_1 (s_1, s_1 + \theta_1 \rho) + P u_1' (s_1, \theta_1 \rho + s_1) \left(1 - \theta_1 - \rho \frac{\partial \theta_1}{\partial s_1} \right) = 0$$

We limit our analysis to the case $u_i\left(\lambda_i, s_i\right) = u_i\left(\lambda_i - s_i\right)$ for some smooth, concave, and increasing function $u_i\left(.\right), i = 1, 2$.[33] To reduce the notation, we denote the partial derivatives by sub-indices (e.g. $\frac{\partial f}{\partial x} = f_x$. So the SP constraint becomes

$$P_{s_1}\left(s\right) u_1\left(\theta_1\left(s\right)\rho\right) + P\left(s\right) u_1'\left(\theta_1\left(s\right)\rho\right)\left(1 - \theta_1\left(s\right) - \rho\theta_{1s_1}\left(s\right)\right) = 0$$

Recall that we defined $w_1\left(x\right) = \frac{u_1'\left(x\right)}{u_1\left(x\right)}$, $\theta = \theta_1$ so that $\theta_2 = 1 - \theta$, and we can rewrite the above into

$$P_{s_1}\left(s\right) + P\left(s\right) w_1\left(\theta\left(s\right)\rho\right)\left(1 - \theta\left(s\right) - \rho\theta_{s_1}\left(s\right)\right) = 0 \tag{5}$$

Similarly, for agent 2 we obtain

$$P_{s_2}\left(s\right) + P\left(s\right) w_2\left(\left(1 - \theta\left(s\right)\right)\rho\right)\left(\theta\left(s\right) + \rho\theta_{s_2}\left(s\right)\right) = 0 \tag{6}$$

Equations (5) and (6) define a system of quasi-linear PDEs for the 2 unknown functions θ and P. Any initial Cauchy data on a non-characteristic curve will uniquely determine its solution.[34] In this case, a curve Γ is non-characteristic, if we can compute the values of all derivatives $\theta_{s_1}, \theta_{s_2}, P_{s_1}, P_{s_2}$ uniquely from the values of P and θ at any point on Γ.

Let Γ be the 0- surplus line, i.e. the line $\rho = 0$ or $s_2 = 1 - s_1$. We will now show two things. First, given our assumptions this curve is always non-characteristic, and second, for any utilities u there is a unique set of initial data for which the problem is a regular, well-posed problem.

In order to show these two claims, first note that from our assumptions on $u_i\left(x\right)$, the function $w_i\left(x\right)$ has a singularity at $x = 0$, since $u_i\left(0\right) = 0$ and $u_i'\left(0\right) > 0$. Moreover, close to $x = 0$, $\frac{u_i'\left(x\right)}{u_i\left(x\right)}$ has a singularity of order 1 .That is, $w_i\left(x\right)$ has the Laurent series expansion around $x = 0$ of the form

$$w_i\left(x\right) = \frac{K_i}{x} + \sum_{n=0}^{\infty} a_{i,n} x^n$$

We will now exploit this as follows. Clearly, the only possible Cauchy data for P has $P\left(s\right)\vert_{\rho=0} = 0$, since otherwise either P_{s_1} or P_{s_2} is unbounded at $\rho = 0$. Along $\rho = 0$ we have $\frac{\partial}{\partial s_2} = -\frac{\partial}{\partial s_1}$ (since $s_2 = 1 - s_1$), so we can express $P_{s_2} = -P_{s_1}$ and $\theta_{s_2} = -\theta_{s_1}$. In order to show that $\rho = 0$ is non-characteristic, we need to show that the system

$$P_{s_1}\left(s\right) + P\left(s\right) w_1\left(\theta\left(s\right)\rho\right)\left(1 - \theta\left(s\right) - \rho\theta_{s_1}\left(s\right)\right) = 0 \tag{7}$$

$$-P_{s_1}\left(s\right) + P\left(s\right) w_2\left(\left(1 - \theta\left(s\right)\right)\rho\right)\left(\theta\left(s\right) - \rho\theta_{s_1}\left(s\right)\right) = 0$$

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[33]The relaxation of this assumption is straight-forward but amounts to cumbersome notation.
[34]See John [1982].

has a unique solution for P_{s_1} and θ_{s_1} along $\rho = 0$.

First suppose that $\theta|_{\rho=0} \in (0,1)$, so that θ and $1 - \theta$ are both strictly positive. Then $\frac{P_{s_1}}{P}$ and $w_1(\theta\rho)$ must be at least of the same order as $\rho \to 0$, meaning that $\frac{P_{s_1}}{P}$ must be at least of the order $\frac{1}{\rho}$ as $\rho \to 0$. Suppose first that $\frac{P_{s_1}}{P}$ is of the order $\frac{1}{\rho^{\tilde{K}}}$, where $\tilde{K} > 1$. Then $P_{s_1} = P_{s_2} = 0$, so $P(s) = 0$ in some neighborhood of $\rho = 0$, which from our system of equations for the derivatives of P implies that P is in fact equal to 0 everywhere, which gives the trivial solution to our system.

So assume that the order $\frac{P_{s_1}}{P}$ is precisely $\frac{1}{\rho}$. Denote by $p_0 = \lim_{\rho \to 0} \frac{\rho P_{s_1}(s)}{P(s)} > 0$. We can multiply the whole system (7) by ρ to obtain, after elimintating the terms that limit to 0, that

$$p_0 + \frac{K_1}{\theta(s)}(1 - \theta(s) - \rho\theta_{s_1}(s)) = 0 \tag{8}$$

$$-p_0 + \frac{K_2}{(1 - \theta(s))}(\theta(s) - \rho\theta_{s_1}(s)) = 0$$

must have a unique solution for p_0 and θ_{s_1} when $\rho \to 0$. We can now express

$$\lim_{\rho \to 0} \theta_{s_1}(s) = \lim_{\rho \to 0} \frac{1}{\rho} \frac{\frac{K_1}{\theta(s)}(1 - \theta(s)) - \frac{K_2}{(1 - \theta(s))}\theta(s)}{\frac{K_1}{\theta(s)} + \frac{K_2}{(1 - \theta(s))}},$$

so that the only possibility for $\theta_{s_1}(s)$ being bounded at $\rho = 0$ is if $\lim_{\rho \to 0} \frac{K_1}{\theta(s)}(1 - \theta(s)) - \frac{K_2}{(1 - \theta(s))}\theta(s) = 0$. Then $\theta(s)$ is locally a constant $\theta(s) = \frac{(K_1 K_2)^{\frac{1}{2}}}{1 + (K_1 K_2)^{\frac{1}{2}}}$, so that $\theta_{s_1}|_{\rho=0} = 0$.

Finally, we have to check $\lim_{\rho \to 0} \theta(s) = 0$ is possible only if $P(s) = 0$ everywhere (the possibility $\lim_{\rho \to 0} \theta(s) = 1$ is treated analogously). So suppose that $\lim_{\rho \to 0} \theta(s) = 0$. Then, in order for P_{s_1} not to explode near $\rho = 0$, the order of $\frac{P_{s_1}}{P}$ again has to be at least 1 so the only possibility is $p_0 = 0$, since otherwise the first equality in (8) wouldn't be satisfied. Thus, the system (7) only admits the trivial solution. [35] $\square$

PROOF OF PROPOSITION 6:

Proof. The quasi-linear system (1) has a solution if and only if $\frac{\partial^2 P(s)}{\partial s_1 \partial s_2} = \frac{\partial^2 P(s)}{\partial s_2 \partial s_1}$. It is a matter of straight-forward calculus to check that when $u_i = C u_j$, where $C > 0$, $\theta = \frac{1}{2}$ is the unique θ that makes this cross-partials condition be fulfilled. We can then plug $\theta = \frac{1}{2}$ into the

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[35] We could try to proceed similarly if either $w_1(x)$ or $w_2(x)$, but not both, have a singularity of the order $x \log x$ at $\rho = 0$. However, then it is impossible to prove that the line $\rho = 0$ is non-characteristic.

equation (1) of Proposition 4 to obtain

$$\frac{\frac{\partial P(s)}{\partial s_i}}{P(s)} = -\frac{u_i'\left(\frac{1}{2}\rho\right)}{2\,u_i\left(\frac{1}{2}\rho\right)}; \; i = 1, 2.$$

By integrating this simple system we obtain the solution for $P(s)$

$$P(s) = \pi \frac{u_i\left(\frac{1}{2}\rho\right)}{u_i\left(\frac{1}{2}\right)}, \; \pi \in (0, 1].$$

Now we apply Theorem 5 which proves the part of the statement for equal utilities.

We proceed similarly to show that the mechanism defined by

$$\theta^* = \frac{\gamma_1}{\gamma_1 + \sqrt{\gamma_2 \gamma_1}}$$

$$P^*(\rho) = \begin{cases} e^{-D_1\theta_1 s_1 - D_2\theta_2 s_2} \rho^{\sqrt{\gamma_1\gamma_2}}; \; s_1 + s_2 \le 1 \\ 0; \; \text{otherwise.} \end{cases},$$

is decent when the utilities are of the form

$$u_i(\lambda_i, s_i) = C_i(\lambda_i - s_i)^{\gamma_i} e^{D_i(\lambda_i - s_i)},$$

$$\gamma_i \in (0, 1], \; D_i \in \left[-\gamma_i - \sqrt{\gamma_i}, \sqrt{\gamma_i} - \gamma_i\right].$$

Now we prove that there for no other utilities there exists a fixed share mechanism. So assume that $u_1 \ne u_2$ and take $\theta_i = \theta_i^* = const$. Then the equation $\frac{\partial^2 P(s)}{\partial s_1 \partial s_2} = \frac{\partial^2 P(s)}{\partial s_2 \partial s_1}$ simplifies to

$$\frac{(u_1'(x_1))^2 - u_1''(x_1)u_1(x_1)}{[u_1(x_1)]^2} = \frac{(u_2'(x_2))^2 - u_2''(x_2)u_2(x_2)}{[u_2(x_2)]^2},$$

which is in fact

$$\left(\frac{u_1'(x_1)}{u_1(x_1)}\right)' = \left(\frac{u_2'(x_2)}{u_2(x_2)}\right)'. \tag{9}$$

If $\theta_i = \frac{1}{2}$ it is easily seen that the equation (9) implies $u_1 = Cu_2$, $C > 0$. So assume that $\theta_i \ne \frac{1}{2}$. Then (9) implies that $\left(\frac{u_i'(x_i)}{u_i(x_i)}\right)' = \bar{h}_i(x_i)$ where $\bar{h}_i(x_i)$ has the property $\bar{h}_i(tx) = K_i g_i(t) \tilde{h}(x)$ - otherwise θ_i would necessarily be a function of ρ. Reversing the roles of t and x this means that $\bar{h}_i(tx) = \bar{h}_j(tx) = \tilde{h}(tx) = \tilde{h}(t)\tilde{h}(x)$. It is easy to see that the only class of functions for which the last equality holds is the class $\tilde{h}(x) = K_i x^h$, where h is a constant. Next, we also get that in order for concavity of u to hold, it has to be that $\lim_{x \to 0} u_i(x) = O\left(x^{-\frac{h}{2}}\right)$. But such u will only be concave in the neighborhood of $x = 0$ if $h \in [-2, 0]$. Integrating

$$\left(\frac{u_i'(x)}{u_i(x)}\right)' = K_i x^h$$

we obtain for $h > -2$ utilities that cannot satisfy the requirement $u_i(0) = 0$. For $h = -2$ we obtain precisely the above class of utilities. $\square$

PROOF OF THEOREM 7:

Proof. We will first show that whenever the decent mechanism exists, it is net-surplus. Then, we will show that the decent mechanism always exists. It is straightforward to check that if $\theta(s_1, s_2) = \theta(\rho)$ and $P(s_1, s_2) = P(\rho)$ then the (1) is equivalent to

$$\rho\theta'(\rho) + \theta(\rho) - \frac{w_1(\rho\theta(\rho))}{w_2(\rho(1 - \theta(\rho))) + w_1(\rho\theta(\rho))} = 0 \tag{10}$$

and

$$\frac{P'(\rho)}{P(\rho)} = w_2(\rho(1 - \theta(\rho)))(\theta(\rho) + \rho\theta'(\rho)). \tag{11}$$

It is important to observe that in our case, the boundary conditions in the proof of the Theorem 5 are constant along the line $\rho = 0$. Hence, we can get the desired initial conditions for (10) from those boundary conditions. On the other hand, we have that at the line $\rho = 0$, i.e. $s_1 + s_2 = 1$, $\frac{\partial P(s_1, s_2)}{\partial s_1} = \frac{\partial P(s_1, s_2)}{\partial s_2} = p_0$, so that the boundary condition for P is also symmetric. So, P will be a symmetric function of s_1 and s_2 in a neighboorhood of $\rho = 0$. Now, at every point s_1, s_2, s.t. $s_1 + s_2 < 1$, the solution to (10) and (11) satisfies the partial differential equation (1). Moreover, any decent mechanism coincides with the solution to (10) and (11) on a non-characteristic curve $\rho = 0$. Hence, by the Cauchy-Kowalevski theorem[36], the decent mechanism depends only on the net surplus. Now we have to show that the net-surplus solution always exists. That is, we have to show that θ and P that solve (10) and (11) remain in $[0, 1]$ for $\rho \in [0, 1]$. We first turn to (10). It is convenient to define $\alpha(\rho) = \rho\theta(\rho)$, so that $\alpha'(\rho) = \rho\theta'(\rho) + \theta(\rho)$. We have to prove that $0 \leq \alpha(\rho) \leq \rho$. Substituting α into (10) we obtain the ODE for α:

$$\alpha'(\rho) - \frac{w_1(\alpha(\rho))}{w_2(\rho - \alpha(\rho)) + w_1(\alpha(\rho))} = 0.$$

Note that $\lim_{\rho \to 0} \alpha(\rho) = 0$, moreover, in the neighborhood of $\rho = 0$, $\alpha(\rho)$ is approximately equal to $\rho \lim_{\rho \to 0} \theta(\rho) = \rho \frac{K_1}{K_1 + \sqrt{K_1 K_2}}$. Thus,

$$0 < \lim_{\rho \to 0} \alpha'(\rho) = \frac{K_1}{K_1 + \sqrt{K_1 K_2}} < 1.$$

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[36] See John [1982].

Similarly, at any point $\rho \in [0,1]$, if $\alpha(\rho) \in (0,1)$, then $\alpha'(\rho) \in (0,1)$. Hence, in the neigborhood of $\rho = 0$, $0 < \alpha(\rho) < \rho$, and by the uniqueness theorem for the ordinary differential equations, $0 < \alpha(\rho) < 1$ everywhere. Turning to (11), it can be rewritten as

$$\frac{P'(\rho)}{P(\rho)} = w_2(\rho - \alpha(\rho))(\alpha'(\rho)),$$

so that by integration, we obtain

$$P(\rho) = \pi\, e^{\int_0^\rho w_2(r - \alpha(r))(\alpha'(r))dr}, \pi > 0.$$

By setting the constant $\pi = e^{-\int_0^1 w_2(r - \alpha(r))(\alpha'(r))dr}$, we obtain the decent mechanism. $\square$

PROOF OF LEMMA 9:

Proof. Denote $L_i(s_i) = \lim_{t \to \infty} \lambda_i(s_i, t)$. The proof is divided into three steps. In step 1 we show that $L_i(1) = 1$ (which holds trivially) and the continuity at 1 imply that $L_i(0) = 0$. In step 2 we show that $L_i(.)$ is a continuous function, hence it attains all values in the interval $[0,1]$. Finally, in step 3 we show that the statement of the lemma is true.

Step 1: $L_i(0) = 0$. Suppose this isn't the case, i.e. $L_i(0) = K > 0$ in equilibrium. Denote by $\lambda_i(0,t)$ such equilibrium strategy of player i, and by $\lambda_j(s_j, t)$ the equilibrium strategy of player j, when his type is s_j. By individual rationality we have that $L_j(1) = 1$. Also by individual rationality, we have that $L_j(s_j)$ is bounded below, i.e. $L_j(s_j) \geq s_j$. Since $L_j(s_j) \leq 1$, these imply that $L_j(s_j)$ is continuous at point $s_j = 1$. From continuity of L_j around $s_j = 1$ we get that there is a positive mass of types $s_j \in [0,1]$ for which $L_j(s_j) > 1 - K$. But then type 0 of agent i could improve her expected payoff by playing λ_i until some large time t', and then lowering her demand to 0, according to some strategy λ_i'. To see this, notice that λ_i and λ_j are continuous and for all s_j, $\lambda_i(0,t)$ and $\lambda_j(s_j, t)$ are non-increasing in t. Thus the support of $f_j(s_j|t)$ is shrinking as time elapses. When t is very large, the support of $f_j(s_j|t)$ will be very close to the ex-post belief when no agreement has been reached. Hence t' is given as the moment when the expected continuation payoff of playing λ_i, conditional on $s_j \leq 1 - K$, is lower than the expected continuation payoff of playing λ_i', conditional on $s_j < 1$. This establishes the contradiction. The same argument shows that $L_i(s_i)$ is continuous in a neighbourhood of the point $s_i = 0$.

Step 2. Assume thus that $L_i(s_i)$ is discontinuous at $\bar{s}_i$, i.e. $L_i(\bar{s}_i) = \hat{l}$ and $\lim_{s_i \searrow \bar{s}_i} = \bar{l}$, where $\bar{l} > \hat{l}$. Then there must exist an $\bar{s}_j$ s.t. $L_j(\bar{s}_j) = 1 - \bar{l}$, and $\lim_{s_j \searrow \bar{s}_j} L_j(s_j) = 1 - \hat{l}$

36

(same argument as in Step 1, and left-continuity of L_i and L_j). Take any $\hat{s}_i > \bar{s}_i$. By continuity of λ_i in t, there exists an M_i s.t. $\lambda_i(\bar{s}_i, t) - \hat{l} < \varepsilon$ for all $t \geq M_i$. Also, notice that $\lambda_i(\hat{s}_i, t) \geq \bar{l}$. Now fix $\varepsilon = \frac{\bar{l} - \hat{l}}{4} > 0$ and take a $t \geq M_i$. Then at t, $\lambda_i(\bar{s}_i, t) < \hat{l} + \varepsilon$ while $\lambda_i(\hat{s}_i, t) \geq \bar{l}$ for all $\hat{s}_i > \bar{s}_i$, contradicting the continuity of λ_i in s_i. This proves that $L_i(s_i)$ has to be right-continuous. By assumption, $L_i(s_i)$ is left-continuous[37], hence it is continuous. In step 1 we proved that $L_i(1) = 1$ and $L_i(0) = 0$, so by Rolle's theorem it attains all values between 0 and 1.

Step 3: $L_i(s_i) = s_i$ for all $s_i \in [0, 1]$. Take an $s_i \in (0, 1)$. By steps 1 and 2, L_i takes all the values in the interval $[0, 1]$ and is continuous (thus measurable), strictly positive on $(0, 1]$. Thus we can define the measure μ_i

$$\mu_i(S) = \int_S L_i(s)\, dm(s) \text{ for any mesaurable } S \subset [0, 1],$$

where $m(.)$ denotes the usual Lebesgue measure. By strict positivity, continuity, and boundedness of $L_i(s_i)$, μ_i is an equivalent measure to m. Now suppose that $L_i(s_i) > s_i$. By equivalence of μ_i to m there exists a positive mass of types s_j s.t. $L_j(s_j) \in (1 - L_i(s_i), 1 - s_i)$. To see this define $B = \{s_j | L_j(s_j) \in (1 - L_i(s_i), 1 - s_i)\}$. Since μ_i and m are equivalent, $m(B) > 0$. Now repeat the same argument as in Step 1 to get a contradiction. Hence indeed $L_i(s_i) = s_i$. $\qquad\square$

PROOF OF LEMMA 10 :

Proof. Fix the type-contingent strategy of player j at some strategy $\lambda_j(.,.)$. Denote by $H_j(s_i, t; \lambda_i)$ the mass of types of player j with whom s_i enters in agreement until time t if he plays the strategy $\lambda_i(.,.)$. Observe that at any t, s.t. $\exists s_j$ with $\lambda_i(s_i, t) + \lambda_j(s_j, t) = 1$, $H_j(s_i, t; \lambda_i)$ is strictly increasing if and only if $\lambda_i(s_i, .)$ is strictly decreasing or $\lambda_j(s_j, .)$ is strictly decreasing in t. This follows from continuity of $\lambda_i(.,.)$ and $\lambda_j(.,.)$ w.r.t. s. Moreover, $H_j(s_i, t; \lambda_i)$ has a jump at t if and only if $\exists s'_j, s''_j$ s.t. $\lambda_j(s_j, t) + \lambda_i(s_i, t) = 1$ for all $s_j \in (s'_j, s''_j)$.

We have to show that $\lambda_i(s_i, t) \geq \lambda_i(s'_i, t)$ for any $s_i \geq s'_i$ and any t s.t. $H_j(s_i, t; \lambda_i)$ is strictly increasing at t (at any s_i, where condition in the statement of the lemma is satisfied,

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[37]Type $\bar{s}_i$ is at $t = \infty$ indifferent between demanding $\hat{l}$ and $\bar{l}$; the former doesn't improve her probability of reaching an agreement since the mass of opposing types with demands between $1 - \bar{l}$ and $1 - \hat{l}$ is 0. However, by an argument similar to the proof of Step 1, we can argue, that she doesn't lose anything by bidding $\hat{l}$, which gives us left-continuity of L_i. Left-continuity of L_i is thus essentially an assumption on how agents resolve their indifference at the horizon.

$H_j(s_i, t; \lambda_i)$ is strictly increasing, and it can have a jump). We proceed by contradiction. Assume there are $s_i > s'_i$ and t' s.t. $\lambda_i(s_i, t') < \lambda_i(s'_i, t')$ and $H_j(s_i, t'; \lambda_i)$ is strictly increasing at t'. Denote by $t_0 = \inf\{t | H_j(s_i, t; \lambda_i) > 0, t < t', \text{ and } \lambda_i(s_i, \tau) < \lambda_i(s'_i, \tau) \text{ for all } \tau \in (t, t')\}$ and by $t_1 = \min\{t | t > t', \lambda_i(s_i, t) = \lambda_i(s'_i, t)\}$. In other words, t_0 is the largest time until which the demands of s_1 and s'_1 are monotonic, and t_1 is the first time after t_1 at which these demands are equal. First, by continuity of $\lambda_i(s_i, .)$ and $\lambda_i(s'_i, .)$ it is clear that $t_0 < t' < t_1$. Moreover, $t_1 < \infty$ since by the previous lemma, $\lim_{t \to \infty} \lambda_i(s_i, t) = s_i > s'_i = \lim_{t \to \infty} \lambda_i(s'_i, t)$, hence, by continuity there exists a $\bar{t} < \infty$ s.t. $\lambda_i(s_i, t) > \lambda_i(s'_i, t)$ for all $t \geq \bar{t}$. Since $H_j(s_i, t'; \lambda_i)$ is strictly increasing at t', it is also clear that $H_j(s_i, t_0; \lambda_i) < H_j(s_i, t_1; \lambda_i)$.

If some type has a lower demand at time t he will have agreed with a larger mass of the opponent's types. In other words, $\lambda_i(s_i, t) \leq \lambda_i(s'_i, t) \Rightarrow H_j(s_i, t; \lambda_i) \geq H_j(s'_i, t; \lambda_i)$ for all t and all s_i and s'_i, which follows from the monotonicity of $\lambda_i(., .)$ and $\lambda_j(., .)$ w.r.t. t. Applying this twice at t_0 and t_1, we get $H_j(s_i, t_0; \lambda_i) = H_j(s'_i, t_0; \lambda_i)$ and that $H_j(s_i, t_1; \lambda_i) = H_j(s'_i, t_1; \lambda_i)$. By construction, we have $\lambda_i(s_i, t) < \lambda_i(s'_i, t)$ for all $t \in (t_0, t_1)$. This implies that $H_j(s_i, t; \lambda_i) \geq H_j(s'_i, t; \lambda_i)$ for all $t \in (t_0, t_1)$.

In equilibrium, $\lambda_i(s_i, .)$ is the optimal strategy for type s_i, and $\lambda_i(s'_i, .)$ is optimal for type s'_i on the interval (t_0, t_1). In particular (from now on we drop parameter λ_i in $H_j(.)$),

$$\int_{t_0}^{t_1} e^{-t} u_i(\lambda_i(s_i, t) - s_i) \, dH_j(s_i, t) \geq \int_{t_0}^{t_1} e^{-t} u_i(\lambda_i(s'_i, t) - s_i) \, dH_j(s'_i, t) \tag{12}$$

and

$$\int_{t_0}^{t_1} e^{-t} u_i(\lambda_i(s'_i, t) - s'_i) \, dH_j(s'_i, t) \geq \int_{t_0}^{t_1} e^{-t} u_i(\lambda_i(s_i, t) - s'_i) \, dH_j(s_i, t). \tag{13}$$

Subtracting these two inequalities, we obtain

$$\int_{t_0}^{t_1} e^{-t} [u_i(\lambda_i(s_i, t) - s_i) - u_i(\lambda_i(s_i, t) - s'_i)] \, dH_j(s_i, t) \tag{14}$$

$$\geq \int_{t_0}^{t_1} e^{-t} [u_i(\lambda_i(s'_i, t) - s_i) - u_i(\lambda_i(s'_i, t) - s'_i)] \, dH_j(s'_i, t).$$

By concavity of $u_i(.)$, and since $\lambda_i(s_i, t) - s_i - (\lambda_i(s_i, t) - s'_i) = \lambda_i(s'_i, t) - s_i - (\lambda_i(s'_i, t) - s'_i) = s'_i - s_i < 0$ we have that $u_i(\lambda_i(s_i, t) - s_i) - u_i(\lambda_i(s_i, t) - s'_i) \leq u_i(\lambda_i(s'_i, t) - s_i) - u_i(\lambda_i(s'_i, t) - s'_i)$, so from this and (14) we obtain

$$\int_{t_0}^{t_1} e^{-t} dH_j(s_i, t) \leq \int_{t_0}^{t_1} e^{-t} dH_j(s'_i, t).$$

Integrate by parts to get $\int_{t_0}^{t_1} e^{-t} dH_j(s_i, t) = H_j(s_i, t_1) e^{-t_1} - H_j(s_i, t_0) e^{-t_0} + \int_{t_0}^{t_1} e^{-t} H_j(s_i, t) \, dt$, and similarly for the right hand side. Now we use $H_j(s_i, t_0) = H_j(s'_i, t_0)$ and $H_j(s_i, t_1) =$

$H_j\left(s_i', t_1\right)$, to obtain

$$\int_{t_0}^{t_1} e^{-t} H_j\left(s_i, t\right) dt \le \int_{t_0}^{t_1} e^{-t} H_j\left(s_i', t\right) dt.$$

But since $H_j\left(s_i, t_1\right) \ge H_j\left(s_i', t_1\right)$ for all $t \in (t_0, t_1)$ the last inequality implies that it must in fact be $H_j\left(s_i, t\right) = H_j\left(s_i', t\right)$ for almost all $t \in (t_0, t_1)$. Now take for example (12), and rewrite it into $\int_{t_0}^{t_1} e^{-t}\left(u_i\left(\lambda_i\left(s_i, t\right) - s_i\right) - u_i\left(\lambda_i\left(s_i', t\right) - s_i\right)\right) dH_j\left(s_i, t\right) \ge 0$. But $u_i\left(\lambda_i\left(s_i, t\right) - s_i\right) - u_i\left(\lambda_i\left(s_i', t\right) - s_i\right) < 0$ since $u_i\left(.\right)$ is increasing and $\lambda_i\left(s_i', t\right) > \lambda_i\left(s_i, t\right)$ for $t \in (t_0, t_1)$, which implies that $\int_{t_0}^{t_1} e^{-t}\left(u_i\left(\lambda_i\left(s_i, t\right) - s_i\right) - u_i\left(\lambda_i\left(s_i', t\right) - s_i\right)\right) dH_j\left(s_i, t\right) < 0$, which is a contradiction. $\qquad\square$

PROOF OF LEMMA 14:

Proof. We fix s_i and economize the notation to write $\tilde{s}_j\left(s_i, t\right) = \tilde{s}_j\left(t\right)$ and $\frac{\partial \tilde{s}_j\left(s_i, t\right)}{\partial t} = \dot{\tilde{s}}_j\left(t\right)$. We write the Hamiltonian

$$\begin{aligned}
H_i(t) &= e^{-t} u_i(\lambda_i(s_i, t) - s_i) f_j(\tilde{s}_j(t))\, \dot{\sigma}_j\left(t\right) - \\
&\quad - \mu(t)\left(1 - \lambda_j(\tilde{s}_j(t), t) - \lambda_i(s_i, t)\right)
\end{aligned}$$

and compute the Euler conditions for the unknown functions

$$\begin{aligned}
\frac{\partial H_i}{\partial \tilde{s}_j} &= e^{-t} u_i(\lambda_i(s_i, t) - s_i) f_j'(\tilde{s}_j)\, \dot{\tilde{s}}_j + \mu \frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial \tilde{s}_j} \\
\frac{d}{dt} \frac{\partial H_i}{\partial \dot{\sigma}_j} &= e^{-t} u_i(\lambda_i(s_i, t) - s_i) f_j'(\tilde{s}_j(t))\, \dot{\tilde{s}}_j + e^{-t} u_i'(\lambda_i(s_i, t) - s_i)\frac{\partial \lambda_i(s_i, t)}{\partial t} f_j(\tilde{s}_j) + \\
&\quad - e^{-t} u_i(\lambda_{i(}s_i, t) - s_i) f_j(\tilde{s}_j) \\
\frac{\partial H_i}{\partial \lambda_i} &= e^{-t} u_i'(\lambda_i(s_i, t) - s_i) f_j(\tilde{s}_j)\, \dot{\tilde{s}}_j + \mu \\
\frac{\partial H_i}{\partial \dot{\lambda}_i} &= 0
\end{aligned}$$

Whence we have the two Euler equations

$$\mu \frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial \tilde{s}_j} - e^{-t} u_i'(\lambda_i(s_i, t) - s_i)\frac{\partial \lambda_i(s_i, t)}{\partial t} f_j(\tilde{s}_j)) + e^{-t} u_i(\lambda_i(s_i, t) - s_i) f_j(\tilde{s}_j) = 0$$

$$e^{-t} u_i'(\lambda_i(s_i, t) - s_i) f_j(\tilde{s}_j)\, \dot{\tilde{s}}_j + \mu = 0$$

From the second Euler equation we can eliminate μ and the density f_j also disappears from the first to obtain the final condition

$$\left(u_i'(\lambda_i(s_i, t) - s_i)\right)\left(\frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial \tilde{s}_j}\, \dot{\tilde{s}}_j + \frac{\partial \lambda_i(s_i, t)}{\partial t}\right) + u_i(\lambda_i\left(s_i, t\right)) - s_i) = 0$$

or equivalently

$$u_i(\lambda_i, s_i) = \frac{\partial u_i(\lambda_i, s_i)}{\partial \lambda_i} \left(\frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial s_j} \frac{d\tilde{s}_j}{dt} + \frac{\partial \lambda_i}{\partial t} \right),$$
$$\text{for } t \geq t_E(s_i).$$

□

PROOF OF PROPOSITION 16:

Proof. The strategy $\bar{\lambda}_i(s_i, \tau)$ maximizes the expected gain, that is

$$\bar{\lambda}_i(s_i, \tau) \in \arg\max \int_{t_i^E(s_i)}^{\infty} e^{-\tau} u_i(\bar{\lambda}_i(s_i, \tau), s_i) f_j(\bar{s}_j(s_i, \tau)) \frac{\partial \bar{s}_j(s_i, \tau)}{\partial \tau} d\tau$$

$$\text{subject to } \bar{s}_j\left(s_i, \tau_i^E(s_i)\right) = 0.$$

Now make a substitution $\delta(t) = e^{-\tau}$, so that $\dot{\delta}(t) dt = -e^{-\tau} d\tau$. Also, defining $\tilde{s}_j(s_i, t) = \bar{s}_j(s_i, \tau)$, we have $\frac{\partial \bar{s}_j(s_i, \tau)}{\partial \tau} = \frac{\partial \tilde{s}_j(s_i, t)}{\partial t} \frac{dt}{d\tau}$. Inserting all this into the above program we see that then $\bar{\lambda}_i(s_i, -\ln(\delta(t)))$ maximizes

$$\int_{t_i^E(s_i)}^{\infty} \delta(t) u_i(\bar{\lambda}_i(s_i, -\ln(\delta(t))), s_i) f_j(\tilde{s}_j(s_i, t)) \frac{\partial \tilde{s}_j(s_i, t)}{\partial t} dt$$

$$\text{subject to } \tilde{s}_j\left(s_i, t_i^E(s_i)\right) = 0;$$

which completes the proof. □

PROOF OF THEOREM 20:

Proof. We have already shown that the regular equilibria of the FD game implement decent rules. What we still need to show is that by taking all possible solutions of the first order condition for the FD game we get all possible decent mechanisms. We will show this by demonstrating that the equilibria of the FD game translate into decent rules via a simple substitution. So take strategies $\lambda_1(s_1, t)$ and $\lambda_2(s_2, t)$ that solve

$$\frac{\partial u_i(\lambda_i, s_i)}{\partial \lambda_i} \left(\frac{\partial \lambda_j(\tilde{s}_j, t)}{\partial s_j} \frac{\partial \tilde{s}_j}{\partial t} + \frac{\partial \lambda_i}{\partial t} \right) = u_i(\lambda_i, s_i) \tag{15}$$

$$\text{s.t.} \lambda_i(s_i, t) + \lambda_j(\tilde{s}_j, t) = 1 \tag{16}$$

Implicitly derive the relationship (16) on t to get

$$\frac{\partial \lambda_j\left(\tilde{s}_j, t\right)}{\partial s_j} \frac{\partial \tilde{s}_j}{\partial t} + \frac{\partial \lambda_i}{\partial t} = -\frac{\partial \lambda_j}{\partial t} \tag{17}$$

Define $Y_j\left(s_j, s_i\right) = \lambda_j\left(s_j, t\left(s_i, s_j\right)\right)$, where $t\left(s_i, s_j\right)$ is defined from the relationship $\lambda_i\left(s_i, t\right) + \lambda_j\left(s_j, t\right) = 1$. Thus $\frac{\partial \lambda_j}{\partial t} \frac{\partial t}{\partial s_i} = \frac{\partial Y_j}{\partial s_i}$, hence

$$\frac{\partial \lambda_j}{\partial t} = \frac{\partial Y_j}{\partial s_i} \frac{1}{\frac{\partial t}{\partial s_i}} \tag{18}$$

Now substitute (17), (16), and (18) into (15) to obtain

$$-\frac{\partial u_i\left(Y_i, s_i\right)}{\partial Y_i} \frac{\partial Y_j}{\partial s_i} \frac{1}{\frac{\partial t}{\partial s_i}} = u_i\left(Y_i, s_i\right) \tag{19}$$

Since $Y_j\left(s_j, s_i\right) + Y_i\left(s_i, s_j\right) = 1$, we have that $\frac{\partial Y_j}{\partial s_i} = -\frac{\partial Y_i}{\partial s_i}$. Now interpret the probability of implementation as the discount due to delay, that is $P\left(s_i, s_j\right) = e^{-t\left(s_i, s_j\right)}$. Hence

$$\frac{\partial P\left(s_i, s_j\right)}{\partial s_i} = -e^{t\left(s_i, s_j\right)} \frac{\partial t\left(s_i, s_j\right)}{\partial s_i} = -P\left(s_i, s_j\right) \frac{\partial t\left(s_i, s_j\right)}{\partial s_i}$$

Thus

$$\frac{\partial t\left(s_i, s_j\right)}{\partial s_i} = -\frac{\frac{\partial P\left(s_i, s_j\right)}{\partial s_i}}{P\left(s_i, s_j\right)}$$

Plugging all of this into (19) we get that $\lambda_i\left(.\right)$ and $\lambda_j\left(.\right)$ satisfy (15) if and only if $Y_i\left(.\right)$, $Y_j\left(.\right)$, and $P\left(s_i, s_j\right)$ satisfy the first order condition

$$-\frac{\partial u_i\left(Y_i, s_i\right)}{\partial Y_i} \frac{\partial Y_i}{\partial s_i} = \frac{\frac{\partial P\left(s_i, s_j\right)}{\partial s_i}}{P\left(s_i, s_j\right)} u_i\left(Y_i, s_i\right)$$

This completes the proof. $\qquad\square$

PROOF OF PROPOSITION 22:

Proof. Observe that the proposed strategies are regular and satisfy Lemmas 8 to 11. Therefore, to show that they constitute an equilibrium of the FD game, it suffices to check that they satisfy (3). Let θ^* be the constant $\theta^* = \frac{\gamma_1}{\gamma_1 + \sqrt{\gamma_1 \gamma_2}}$, and denote $h\left(t\right) = e^{-\frac{t}{\sqrt{\gamma_1 \gamma_2}}}$. We check (3) for λ_1, the calculus for λ_2 is analogous. Now

$$\begin{aligned}
\tilde{s}_2\left(s_1, t\right) &= 1 - s_1 - h\left(t\right) \\
\frac{\partial \lambda_2\left(\tilde{s}_2, t\right)}{\partial s_2} &= 1, \quad \frac{\partial \tilde{s}_2}{\partial t} = -\dot{h}\left(t\right) \\
\frac{\partial \lambda_1}{\partial t} &= \theta^* \dot{h}\left(t\right)
\end{aligned}$$

so by substituting λ_1 into the first order condition, we get

$$\gamma_1 \left(\lambda_1 - s_1\right)^{\gamma_1 - 1} \left(-\dot{h} + \theta^* \dot{h}\right) = (\lambda_1 - s_1)^{\gamma_1}$$

Noticing that $\lambda_1 - s_1 = \theta^* h$. This simplifies into

$$\frac{\theta^*}{\gamma_1 \left(1 - \theta^*\right)} = -\frac{\dot{h}}{h}.$$

Deriving the analogous expression from (3) for agent 2 yields

$$\frac{(1 - \theta^*)}{\gamma_2 \theta^*} = -\frac{\dot{h}}{h}.$$

Substituting h and θ^*, it is immediate that (3) holds. $\qquad\square$

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